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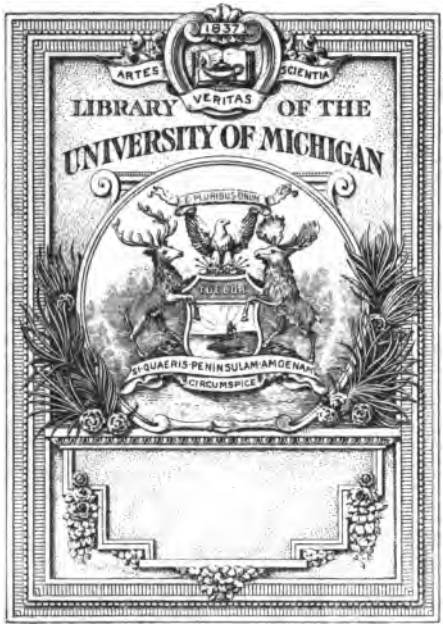
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WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

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CONTENTS.

Mathematical Papers, &c.

No.		Page
70.	Note on Random Lines. By W. S. B. Woolhouse, F.R.A.S. ...	33
71.	Note on the Solution of Question 2483. By Artemas Martin...	37
72.	Note on Prime Numbers. By A. De Morgan	53
73.	Note on Analagmatic Pavements, Quest. 2552. By Professor Sylvester.....	112

Solved Questions.

1971. In a given tetrahedron, to find a set of rectangular coordinate axes, such that each axis shall pass through a pair of opposite edges 68

2100. 1. From one end O of a fixed diameter OD of a circle, two chords OA and OB are drawn such that the rectangle OA . OB = a constant (k^2) ; show that the envelope of the line AB is a circle with centre O and radius a third proportional to OD and k.

2. On a chord OA of a circle OAD, a point C is taken such that the sum or difference of the squares on OC and OA = a constant (k^2) ; show that the envelope of the perpendicular drawn to OC through C is a conic having O as centre 97

2267. If ∇ represent the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and if a', b', c', f', g', h', be the minors which respectively are the coefficients of a, b, c, f, g, h, in the expansion of ∇ : then show that the coordinates (x, y) of the foci of $(abcfgh)\ (xy1)^2=0$, are given by

$$2c'x-2g' = \nabla^{\frac{1}{3}}\left\{(a-b+2hi)^{\frac{1}{3}}+(a-b-2hi)^{\frac{1}{3}}\right\},$$

and $$2c'yi-2f'i = \nabla^{\frac{1}{3}}\left\{(a-b+2hi)^{\frac{1}{3}}-(a-b-2hi)^{\frac{1}{3}}\right\},$$

where i represents $\sqrt{-1}$. Also show that $(c'x-g')\ (c'y-f') = \nabla h$ is the equation to an hyperbola concentric with the given conic and passing through its four foci 78

2334. 1. Construct a triangle of given species, having its vertices on three concentric circles.

No. Page

2. The triangle being equilateral, what relation exists between the radii when only one such triangle can be so constructed? 98

2348. In the system of conics touching two lines at fixed points, the focal cubic cuts a conic in points such that its normals at these points pass through the point of intersection of the two lines. 28

2364. Through each pair of the angles of a triangle circles are described so that the internal segments (in reference to the triangle) are similar. Find the locus of the intersection of the radical axes of the three circles. 29

2382. Given a quartic curve Q, and a point on it P; suppose that from p there are drawn the ten tangents to Q, omitting that at P, touching Q in ten points (t) and meeting it again in ten points (r). The t points, we know, lie on a cubic K; prove that the r points lie on a quartic U, with a point of osculation at p, touching K at p (counting for four points of intersection), and in four other points 67

2383. A and B are fixed points with regard to a conic, ACD a variable straight line passing through A and cutting the curve in C, D. The polar of A meets BC, BD in E, F; show that DE and CF meet in a fixed point G, and that ABG is a straight line 81

2407. Show (1) that the average distance of all the pairs of points that can be taken within a circle, in parts of the radius, is $\frac{128}{45\pi}$; and (2) that the same average for a sphere is $\frac{36}{35}$ 76

2413. Three planes at right angles to each other, and intersecting in a point A, touch an ellipsoid whose centre is B. Prove that the connector AB and the normals at the three points of contact are generators of a hyperboloid. 37

2414. 1. The number of ways in which n things, all different, can be allotted to r different persons, blank lots being admissible, is r^n.

2. If blank lots be *not* admissible, the number of ways is the coefficient of x^n in the expansion of $(\epsilon^x - 1)^r \lfloor n$.

3. The number of ways in which n things, all alike, can be allotted to r different persons, blank lots being admissible, is the number of combinations of $n+r-1$ things taken $r-1$ at a time.

4. If blank lots be *not* admissible, the number of ways is the number of combinations of $n-1$ things taken $r-1$ at a time. 50

2425. A heavy cone stands inverted on its vertex, its axis making an angle α with the horizon; find the magnitude and direction of the least power which, when applied at the centre of its base, will just sustain the cone in that position..... 66

2426. If F_n denote $\frac{1}{a+a^3+\ldots+a^{2n-1}} \frac{a^2+a^4+\ldots+a^{2n}}{}$, prove that, in general,

No. Page

(1) ..... $F_n > \frac{n+1}{n} + \left(\sqrt{a} \quad \frac{1}{\sqrt{a}}\right)^2$,

(2) ..... $F_n > \frac{1}{2} \cdot \frac{n+1}{n}\left(a+\frac{1}{a}\right)$, (3) ..... $\frac{n}{n+1} F_n$

increases with n. 31

2428. Prove that

(1)... $\frac{\Gamma(mx)}{\Gamma(nx)}\ (x=0) = \frac{n}{m}$;

(2)... $\frac{\Gamma(n)}{\Gamma(2n)} - \frac{\Gamma(n+2)}{\Gamma(2n+2)} \cdot \frac{\pi^2}{1.2} + \frac{\Gamma(n+4)}{\Gamma(2n+4)} \cdot \frac{\pi^4}{1.2.3.4} - \&c. = 0.$ 43

2430. A cycloid revolving about its axis generates a solid which moves in the direction of the axis through a fluid with a given velocity; compare the resistances in the two cases when the vertex and the base are respectively foremost. 30

2453. The total number of partitions of n different things (*i.e.* the number of ways in which n different things can be distributed into 1, 2, 3, ... or n indifferent parcels) is the coefficient of x^n in the expansion of $\lfloor n \cdot \epsilon^{\epsilon^x - 1}$ 62

2461. If a curve of the nth degree be translated in one direction, the centre of mean distances of the points common to any two positions of the curve lies always on a straight line. 48

2465. If the tangent drawn to the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, at one end of a chord whose pole is (h, k), be represented by $a^2yy' + b^2xx' = a^2b^2$, find the equation to the tangent at the other end of the polar 66

2474. 1. If $\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 = 0$, show that, when n is any positive integer,

$$\left(\cos\theta \cdot \frac{d}{dx} + \sin\theta \cdot \frac{d}{dy}\right)^n = \left(\cos n\theta \cdot \frac{d}{dx} + \sin n\theta \cdot \frac{d}{dy}\right)\left(\frac{d}{dx}\right)^{n-1};$$

and hence

2. Prove the theorem (due to Rankine and Stokes) that at every multiple point in a plane curve which fulfils the condition $\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} = 0$, the branches make equal angles with each other 99

2482. A quadrilateral circumscribes an ellipse, the eccentric angles of the points of contact being $\alpha, \beta, \gamma, \delta$. Prove that the condition that a pair of opposite vertices should lie on a confocal conic may be thus expressed:—

$a^2(\cos A + \cos B + \cos C + \cos D)(\cos A + \cos B - \cos C - \cos D) \times$
$(\cos C + \cos A - \cos B - \cos D)(\cos B + \cos C - \cos A - \cos D) =$
$b^2(\sin A + \sin B + \sin C + \sin D)(\sin A + \sin B - \sin C - \sin D) \times$
$(\sin C + \sin A - \sin B - \sin D)(\sin B + \sin C - \sin A - \sin D;$

where $A = \frac{1}{2}(-\alpha+\beta+\gamma+\delta)$, $B = \frac{1}{2}(\alpha-\beta+\gamma+\delta)$,
$C = \frac{1}{2}(\alpha+\beta-\gamma+\delta)$, $D = \frac{1}{2}(\alpha+\beta+\gamma-\delta)$ 65

No. Page

2485. In an acute-angled scalene triangle to inscribe the rectangle of minimum diagonal, having one of its sides coincident with a side of the triangle 109

2486. Find a point on an ellipse such that the normal produced to meet the curve again may cut off a maximum area. 56

2497. Show that if $a, b, c \ldots h$ denote

$$\frac{1}{1.2}\cdot\frac{d^2y}{dx^2},\ \frac{1}{1.2.3}\cdot\frac{d^3y}{dx^3},\ \frac{1}{1.2.3.4}\cdot\frac{d^4y}{dx^4},\ \ldots\ \frac{1}{1.2\ldots9}\cdot\frac{d^9y}{dx^9},$$

the general differential equation of

$a_1x^3 + a_2x^2y + b_1xy^2 + b_2y^3 + a_3x^2 + dxy + b_3y^2 + c_1x + c_2y + c_3 = 0$ is

$$\begin{vmatrix} a, & b, & c, & a^2 & , & 0 & , & 0 \\ b, & c, & d, & 2ab & , & a^2 & , & 0 \\ c, & d, & e, & 2ac+b^2 & , & 2ab & , & a^3 \\ d, & e, & f, & 2ad+2bc & , & 2ac+b^2 & , & 3a^2b \\ e, & f, & g, & 2ae+2bd+c^2 & , & 2ad+2bc & , & 3a^2c+3ab^2 \\ f, & g, & h, & 2af+2be+2cd, & & 2ae+2bd+c^2, & & 3a^2d+6abc+b^3 \end{vmatrix} = 0 \quad 47$$

2499. ABCD is a quadrilateral; E, F, G, H the middle points of AB, BC, CD, DA; and $m_1, m_2, \ldots m_8$ the middle points of AE, EB, BF, ... HA. Show that the sum of the squares of the 28 lines joining every two of the points $m_1, m_2, \ldots m_8$ is

$$4(AC^2 + BD^2) + AB^2 + BC^2 + CD^2 + DA^2 \ldots\ldots 67$$

2503. Prove that the number of normals that can be drawn from a point in space of p dimensions to a (hyper)$^{p-3}$ surface (*i.e.*, what corresponds to a *surface* in space of three dimensions) of the nth degree is $\frac{n}{n-2}\left\{(n-1)^p - 1\right\}$ 100

2506. Determine the two positions of the line whose chord of intersection with an assigned one of three given circles shall be cut harmonically by the other two 59

2511. If on an ordinary chessboard we compare any two rows or any two columns, squares opposite one another will be all of unlike or all of like colours according as the number of rows or columns intervening is even (zero included) or odd.

1. Show that it is possible to paint the squares in a manner such that, on comparing any two rows or columns, four of the squares opposite one another shall have like and the remaining four unlike colours.

2. Extend the construction to the case of a chessboard containing 2^4 (instead of eight) squares in each direction..... 74

2514. Let a given volume of space, having a convex boundary *of any form whatever*, be referred to its centre of gravity and the principal axes of rotation, and let M denote the total volume, and h, k, l what may be called compressed radii of gyration round the axes, so that

$$M.h^2 = \iiint x^2\,dx\,dy\,dz,\ M.k^2 = \iiint y^2\,dx\,dy\,dz,\ M.l^2 = \iiint z^2\,dx\,dy\,dz;$$

then

1. If *four* points be taken at random within the given vo-

No. Page

lume, the average square of the volume of the tetrahedron which connects them is $\frac{2}{3}h^2k^2l^2$.

2. If *six* points be taken at random within the given volume as the corners of a hexahedron, the probability that two of the points shall be re-entrant is $\frac{10h^2k^2l^2}{M^2}$. 24

2518. 1. Let P be a point on a given plane curve (A), O a fixed point, and Q the vertex of an equilateral hyperbola, whose centre is O, and which touches the given curve at P. Prove that the tangent at Q to the curve which is the locus of Q makes with OQ an angle equal to that made with OP by the tangent to (A) at P.

2. The curve (B), which is the inverse of the locus of Q with respect to the origin O, will be reciprocal to the given curve (A); that is to say, if (B) be regarded as the given curve, the original curve (A) will be derived from it in precisely the same manner as it has been derived from (A).

[A conic section referred to its centre, and an ellipse of Cassini, are reciprocal curves of this kind.] 38

2519. Given the six distances between four points in a plane; find the condition that one of them should be within the triangle formed by the other three. 19

2521. Two lines in space intersect a plain conic; shew that the ruled surface determined by the three as directors is a quadric. 63

2523. If a, b, c be three closed areas situated in such a manner that no straight line can be drawn meeting more than two of them, the average area of all triangles formed by joining three points taken at random in A, B, C respectively, is equal to the area of the triangle formed by joining the centres of gravity of A, B, C.

2. If A, B, C, D be four closed volumes such that no plane can meet more than three of them, the average volume of the tetrahedron formed by joining four points taken at random in A, B, C, D respectively is equal to the volume of the tetrahedron formed by joining the centres of gravity of A, B, C, D respectively. 46

2532. If within a given enclosed area three points be taken at random as the apices of a triangle, the average of the square of the area of all such triangles will be reduced to one-third the value if one of the points be fixed at the centre. Also, if within a given volume of space four points be taken at random as the apices of a tetrahedron, the average of the square of the volume of all such tetrahedra will be reduced to one-fourth the value if one point be fixed at the centre. Show that this theorem is true when the enclosed area or volume of space is *of any form whatever*. 24

2535. The ends of any diameter of an ellipse are joined with one of the foci; prove that the angles made by the joining lines with the major axis are the eccentric angles corresponding to the ends of a focal chord. 32

2548. Given $\cos^2 x \cos^2 a = \sin^2 a \cos^2 a - 4 \cos x \sin (x-a) \sin^2 a$; it is required to find the angle x. 30

No. Page

2551. Given the partial differential equation

$$\frac{d^2z}{dy^2} - a^2\frac{d^2z}{dx^2} + \beta\frac{dz}{dy} + \gamma z = 0 \quad \text{.....(I)},$$

wherein a, β, and γ are functions of x only. Show that when the condition

$$\gamma - \frac{1}{4}\left(\frac{da}{dx} + \beta\right)^2 - \frac{a^2}{2}\frac{d}{dx}\left\{\frac{1}{a}\left(\frac{da}{dx} + \beta\right)\right\} = 0 \quad \text{.....(II)}$$

is satisfied, then in general $z = \sqrt{a}\,\epsilon^{\int \frac{\beta}{2a}dx}\,\phi\left(y + \int\frac{dx}{a}\right)$,

where ϕ is an arbitrary function. Show also that, when β is constant and (II) satisfied, then the integral of (I) is

$$z = \sqrt{a}\,\epsilon^{\int \frac{\beta}{2a}dx}\,\phi\left(y + \int\frac{dx}{a}\right) + k\epsilon^{-\int \frac{\beta}{2a}dx}\,\psi\left(y - \int\frac{dx}{a}\right),$$

where ψ is another arbitrary function, and the arbitrary multiplier k is only inserted for convenience 81

2556. Four solids, *having any forms whatever*, being given, show how to determine sets of six points fixed in each, such that if the four solids be placed in any positions in space and four points be taken arbitrarily, one in each solid, as the apices of a tetrahedron, the average square of volume in respect of all such tetrahedra shall always be equal to that of the 1296 tetrahedra formed by connecting all the groups obtainable from the four invariable sets of six points. 24

2561. 1. A point is taken at random on each side of an equilateral triangle; show that the chance that these points form an acute-angled triangle is $1 - \frac{9}{4}\log\left(\frac{3}{8}\epsilon\right)$.

2. Find also the chance that a circle passing through the three points cuts each side of the triangle (not produced) in a second point. 21

2568. Express the roots of the equation

$$a^6x^6 + 18a^4(bd - c^2)x^4 + 81a^2(bd - c^2)^2x^2 + 27d^2(a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd) = 0$$

in terms of α, β, γ, the roots of the equation

$$ax^3 + 3bx^2 + 3cx + d = 0. \quad \text{.....} \quad 44$$

2571. The equation whose roots are the ratios of the roots of $ax^3 + 3bx^2 + 3cx + d = 0$ is

$$a^2d^2x^6 + (3a^2d^2 - 9abcd)x^5 + (6a^2d^2 - 45abcd + 27ac^3 + 27b^3d)x^4 + (7a^2d^2 - 54abcd + 54ac^3 - 81b^2c^2 + 54b^3d)x^3 + (6a^2d^2 - 45abcd + 27ac^3 + 27b^3d)x^2 + (3a^2d^2 - 9abcd)x + a^2d^2 = 0.$$

Prove this, and show that its discriminant is

$$a^4d^4(ad - 9bc)^2(b^3d - ac^3)^4(a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd)^3;$$

and account *à priori* for the circumstance that, while in the equation b and c rise to the third power, a and d only reach the second. 45

2577. Show that the average area of all the ellipses that can be drawn touching a semicircle and its diameter symmetrically, in parts of the area of the ellipse, is $\frac{2}{15}(2 + \sqrt{2})$, or .45523..., if (1) all possible positions of the centre of the ellipse be sup-

No. Page

posed equally probable, and $(2-\sqrt{2})$, or ·58579..., if (2) all distances of the chord of contact from the centre of the semicircle be supposed equally probable 84

2579. If a line and circle are given, and Q is a point on the tangent to the circle at P so that PQ is bisected by the line, the locus of Q is a quartic with three double points, each branch having an inflexion as it passes through a double point. If the line touches the circle, the quartic becomes the line and a Cissoid of Diocles.

2. Reciprocally, find the envelope of the chord of intersection of a central conic with its circles of curvature. What is it in the parabola? 91

2584. AQM is a semicircle, QC the vertical radius, AB a chord cutting QC in L so that $QC = n \cdot CL$; find the ratio between the triangles BCL and ACL. 32

2585. Four points are taken at random on the sides of a quadrilateral; what is the chance of the lines *joining* them intersecting? 38

2587. Writing $F(z)$ for $a_0 + a_1 z + \frac{a_2 z^2}{1.2} + \frac{a_3 z^3}{1.2.3}$ + &c., show that the coefficient of $x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$ in $F(x_1 - x_2) \cdot F(x_2 - x_3) \ldots$ $\ldots F(x_m - x_1)$ is an invariant, or the leading term of a covariant relative to the function formed by taking the terms of $F(z)$ involved in the coefficient and substituting binomial for exponential coefficients 42

2591. Let AA_1, BB_1, CC_1 be the bisectors of the triangle ABC, intersecting in G_1; let AA_2, BB_2, CC_2 be the perpendiculars from the angular points on the opposite sides, intersecting in G_2; and let B_1C_1, B_2C_2 intersect in P; C_1A_1, C_2A_2 in Q; and A_1B_1, A_2B_2 in R. Then AP, BQ, CR will all be perpendicular to G_1G_2; and the triangle PQR will "circumscribe" the triangle ABC. Let A_i, B_i, C_i be the feet of the interior bisectors of the angles A, B, C, and A_e, B_e, C_e the feet of the exterior bisectors: then the six straight lines PA_i, QB_i, RC_i, PA_e, QB_e, RC_e pass three and three through four points *which are the points of contact of the nine-point circle with the inscribed and escribed circles*... 51

2604. Prove that, in the third system of pulleys, (these being supposed n in number, of equal weight w, and equal diameter $2a$,) the distance of the centre of the parallel tensions of the cords from the first cord is equal to

$$\frac{4\{(n-1)2^{n-1}+1\}W + n\{(n-3)2^n + n + 3\}w}{2(2^n - 1)W} a. \ldots\ldots$$ 55

2605. A horizontal beam AB, supported but not fixed at both ends A and B, is traversed from end to end by a moving load W distributed uniformly over a segment of it of constant length PQ. Show that the greatest bending moment M at any point X of the beam corresponds to the position of the load in its

No. Page

passage over the point for which PX : QX = AX : BX, and has for value

$$M = \frac{W}{AB^2} \cdot AX \cdot BX \cdot (AB - \tfrac{1}{2}PQ) \quad \ldots\ldots\ldots\ldots \quad 60$$

2607. In three straight lines given in position, to determine the positions of the vertices of a triangle of given or minimum magnitude, and of given species. 63

2609. Given three conics passing through the same four points; and on the first a point A, on the second a point B, and on the third a point C. It is required to find, on the first a point A′, on the second a point B′, and on the third a point C′, such that the intersections of the lines

A′B′ and AC, A′C′ and AB, lie on the first conic;
B′C′ and BA, B′A′ and BC, lie on the second conic;
C′A′ and CB, C′B′ and CA, lie on the third conic. 17

2612. If three points P, Q, R be taken at random, one on each of the sides BC, CA, AB, of the triangle ABC, the chance that the area of the triangle PQR shall be greater than $(n+\frac{1}{4})$ of the area of the triangle ABC (n being positive and $< \frac{3}{4}$) is

$$\frac{3-4n}{4} - \frac{12n+1}{4} \log \frac{4}{4n+1} + 8n^{\frac{3}{2}} \left(\tan^{-1} \frac{1}{2n^{\frac{1}{2}}} - \frac{\pi}{6} \right) \quad \ldots\ldots \quad 109$$

2614. Three points are taken at random, one on each of three faces of a tetrahedron, show that the chance that the plane passing through them cuts the fourth face is $\frac{1}{4}$. 54

2620. 1. A rigid body is rotating round a fixed axis; suddenly this axis becomes free, and a parallel axis becomes fixed elsewhere in the body. Find the new initial angular velocity, and the locus of the new axis if the angular velocity is unchanged.

2. A right prism, whose base is a regular polygon of n sides, falls (without sliding) down a rough inclined plane, the contact being supposed to take place only at the corners; find its angular velocity just before the mth impact.

3. Hence deduce the law of motion of a cylinder or sphere down a rough inclined plane. 22

2624. If ABC, A″B″C″ be two triangles, and T any point such that the area of the triangle $(\alpha\beta\gamma)$ formed by the intersections (α, β, γ) of TA″, TB″, TC″ respectively with BC, CA, AB, is constant; then the locus of T is a cubic having only one arbitrary parameter. 79

2626. To draw through a given point within a circle a chord which shall be divided at the point in extreme and mean ratio. 23

2627. How many different signals can be made with a set of n different flags, using r of them at a time, and arranging them on s masts, where $n > r > s$? 59

2630. Show that the degree of the locus of the foci of a system of conics, subject to any four conditions, is three times as great as that of the locus of their centres 63

2631. A conic F has double contact with one of a set of conics having double contact with one another. Show that the

No. Page

chords of intersection of F with each of the set will intersect on the common chord of contact of the whole set 78

2639. The direction angles of three lines through the origin are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$; required the equation of the cone of revolution of which they are sides 62

2641. Find the value of the double integral

$$\iint \frac{dx\,dy}{\left\{(x^2+y^2+a^2-b^2)^2 - 4(a^2-b^2)x^2\right\}^{\frac{1}{2}}},$$

subject to the condition $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$. 20

2646. A cube being cut by any plane, what is the chance that the section is a hexagon? 36

2647. In the equation $x^3 - qx + r = 0$ it is known that q and r both lie between $+1$ and -1; assuming all values between these limits to be equally probable, prove that the chance of all the roots of the equation being real is $\frac{3}{4\pi}\sqrt{3}$. 43

2661. Given five conics, it is of course possible in an infinity of ways to determine five constants a, b, c, &c., so that

$$aU_1 + bU_2 + cU_3 + dU_4 + eU_5$$

may be either a perfect square L^2 or the product of two factors MN. Prove that the lines L all touch a conic, and that the lines M, N are conjugate with regard to that conic; from which it follows that, if M be given, N passes through a fixed point. 19

2662. Prove that

$$\frac{\sin n\alpha \sin(\beta-\gamma) + \sin n\beta \sin(\gamma-\alpha) + \sin n\gamma \sin(\alpha-\beta)}{\sin(\gamma-\beta) + \sin(\beta-\alpha) + \sin(\alpha-\gamma)} = 2\Sigma(p\alpha+q\beta+r\gamma);$$

p, q, r being positive integers whose sum is n, no two being zero, and the coefficient when one is zero being 1 instead of 2. 41

2669. If the diagonals d, d' of any plane quadrilateral be divided at their point of intersection in the ratios $p : p'$ and $q : q'$ respectively, and θ be the angle between them, then (1) the square of the third diagonal will be equal to

$$\frac{p^2p'^2(q^2-q'^2)^2d^2 + q^2q'^2(p^2-p'^2)^2d'^2 + 2pp'qq'(p^2-p'^2)(q^2-q'^2)dd'\cos\theta}{(pq-p'q')^2(pq'-p'q)^2};$$

and (2) the line which divides these diagonals in the ratios $m : m'$ and $n : n'$ respectively will divide the third diagonal in the ratio $\frac{mnp'q' - m'n'pq}{p'q' - pq} : \frac{m'npq' - mn'p'q}{pq' - p'q}$ 39

2675. If two triangles lie in perspective, prove that a cubic curve may be drawn through the six angles, such that for each triangle the tangents at its angles shall all three meet together on the curve. 35

2676. Let the sides of the triangle of reference subtend at a point $(\alpha', \beta', \gamma')$ angles $(\theta, \theta', \theta'')$. Then

$$\sin\theta\left(\frac{\alpha}{\alpha'}\right)^{\frac{1}{2}} + \sin\theta'\left(\frac{\beta}{\beta'}\right)^{\frac{1}{2}} + \sin\theta''\left(\frac{\gamma}{\gamma'}\right)^{\frac{1}{2}} = 0$$

No. Page

is a conic touching the three lines and having $(\alpha', \beta', \gamma')$ for focus.

Also $\tan \frac{1}{2}\theta \left(\frac{\alpha'}{\alpha}\right) + \tan \frac{1}{2}\theta' \left(\frac{\beta'}{\beta}\right) + \tan \frac{1}{2}\theta'' \left(\frac{\gamma'}{\gamma}\right) = 0$

is a conic circumscribing the triangle and having $(\alpha', \beta', \gamma')$ for focus.

If we denote by λ, μ, ν the perpendiculars drawn from the vertices of the triangle of reference on any tangent, and ρ, ρ', ρ'' the distances of $(\alpha', \beta', \gamma')$ from the vertices; then

$$\sin \theta \left(\frac{\rho}{\lambda}\right) + \sin \theta' \left(\frac{\rho'}{\mu}\right) + \sin \theta'' \left(\frac{\rho''}{\nu}\right) = 0$$

is the tangential equation of the first of these conics, and

$$\sin \tfrac{1}{2}\theta \left(\frac{\lambda}{\rho}\right)^{\frac{1}{2}} + \sin \tfrac{1}{2}\theta' \left(\frac{\mu}{\rho'}\right)^{\frac{1}{2}} + \sin \tfrac{1}{2}\theta'' \left(\frac{\nu}{\rho''}\right)^{\frac{1}{2}} = 0$$

is the tangential equation of the second. 34

2680. If A and B be two points taken at random within a closed convex figure, and the line AB be produced to meet the boundary in P and Q, prove that the chance that the three segments PA, AB, BQ are such as would form a triangle is $\frac{1}{4}$. 90

2682. If circles be described on a pair of opposite sides of a square; prove geometrically that

1. The polars of any point on either diagonal with respect to the two circles meet on the other diagonal.

2. The four tangents from the point form a harmonic pencil 101

2687. If normals to circumscribing conics at the vertices of a triangle meet in a point, and if also normals to inscribed conics at the points of contact meet in a point; show that the locus of this point is the same in both cases 58

2690. If four points on a plane be the extremities of the two straight lines a, a' or of the pair b, b', or of the pair c, c'; then, a, b, c being the sides of a triangle, prove that

$$(a^2a'^2 + b^2b'^2 + c^2c'^2)\ (a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)$$
$$= 2\,(a^4a'^2 + a'^4a^2 + b^4b'^2 + b'^4b^2 + c^4c'^2 + c'^4c^2) + a^2b^2c^2 + a^2b'^2c'^2 + a'^2b^2c'^2 + a'^2b'^2c^2. \quad 57$$

2694. A rope without weight passes over a smooth pulley, having a monkey at one end and a weight equal to the monkey at the other. The monkey suddenly runs up the rope to a height h with a uniform velocity; what is the resulting motion of the weight? 53

2695. If a given finite straight line of length a be divided at random in two points, the chance that the product of the three parts shall exceed $\frac{1}{108}a^3$ is

$$\frac{4}{9}\int_{\frac{1}{3}\pi}^{\frac{5}{6}\pi} \sin \tfrac{1}{2}\theta\,(1 + 2\cos 3\theta)^{\frac{1}{2}}\,d\theta \quad \ldots\ldots 58$$

2696. Let $G = 1 + \theta + \theta^2 + \theta^3$ be any group of powers of a substitution made with the elements 1234. Let H_1 be any asymmetric function of $x_1x_2x_3x_4$. And let $\theta; H_1$ or $G; H_1$ be the symbol

No. Page

of the result of operation on H_1 by θ or by the whole group G. Thus $G ; H_1$ is an algebraic sum of four values of H_1.

1. There is a substitution ϕ such that

$$\left\{(G ; H_1^i)^j + \phi ; (G ; H_1^i)^j\right\}^k$$

is a function which has not more than three values, all given with G and H_1, whatever be the integer exponents i, j, k.

2. Hereby the algebraic solution of the general quintic equation is attainable at the cost of few words 70

2697. Let I be a point of inflexion on a cubic. From I draw tangents meeting the cubic in P, Q, R; and from P draw tangents meeting it in A, B, C, D. Prove that the three opposite pairs of sides of the tetragon ABCD will respectively intersect in I, Q, R. 53

2702. The locus, in areal coordinates, of the centre of an ellipse, of constant area (πn^2), circumscribing a triangle, is

$$(x+y+z)\,(x+y-z)\,(x-y+z)\,(-x+y+z) = \frac{4x^2y^2z^2}{n^4} \quad \ldots\ldots 77$$

2704. A central conic ($a^2y^2 \pm b^2x^2 = a^2b^2$) is turned about its centre in its own plane through a right angle; prove that the locus of the intersection of any tangent to the given conic with the same line in its new position is one of the two quartics

$$(x^2+y^2)^2 = a^2\,(x \mp y)^2 \pm b^2\,(x \pm y)^2 \quad \ldots\ldots 111$$

2718. Find *in plano* the locus of a point P, such that from it two given points A, B, and two given points, C, D subtend equal angles 106

2721. Given any convex area Ω, let C be the length of any chord drawn across the area, and let the position of C be defined by (p, θ), the perpendicular on it from a fixed pole inside Ω and the inclination of p to a fixed axis; then $\iint C^3\,dp\,d\theta = 3\Omega^2$, the integration extending to all possible positions of C 89

2732. An epi- or hypo-cycloid is pushed through a very short fixed tube, so as to remain in one plane, show that the locus of its centre is an ellipse 96

2733. Show that the average area of a series of triangles, equably distributed over the surface of an ellipse so as to have each one side parallel to a given line, in parts of the area of the ellipse, is $\frac{256}{525\pi^2}$ 92

2737. Find *in solido* the locus of a point P, such that from it two given points, A, C, and two given points B, D, subtend equal angles. 108

2738. The equation representing the transmission of sound parallel to the axis of a tube, when formed without reference to any previously assumed law of pressure, is $\frac{d^2y}{dt^2} + \frac{1}{D} \cdot \frac{dp}{dx} = 0$, where x is the ordinate of the point of rest of a particle, y the same ordinate at the time t, p the pressure at the same time, and D the density of equilibrium. Show that the equation is

No. Page

satisfied by the values of the pressure, velocity, and density derived from the three following equations, viz.:

$$p = -\frac{a^2}{\rho} + \chi\left(v + \frac{a}{\rho}\right), \quad v + \frac{a}{\rho} = \phi\left(x + \frac{a-\chi'}{D}t\right),$$

$$\frac{1}{\rho} - f\left(v + \frac{a}{\rho}\right) = \psi\left(x - \frac{a}{D}t\right),$$

where a is an arbitrary constant; χ, ϕ, and ψ are arbitrary functions; and f is a function whose form is given by the equation

$$f(z) = \int \frac{dz}{2a - \chi'(z)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 86$$

2743. Show that if p be a prime number and m and n any positive integers, the highest power of p contained in $\frac{\Pi(m+n)}{\Pi(m)\ \Pi(n)}$ may be obtained by expressing $m+n$ and either m or n in the scale of p; the number of times that it would be necessary to borrow in subtracting the latter number from the former being the index of the power of p required 88

2748. If a circular cubic with a double point O be cut by a circle in four points A, B, C, D; and if OA, OB, OC, OD cut the circle again in E, F, G, H; show that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O 105

2756. Show that an infinite number of triangles can be described such that each has the same circumscribing, nine-point, and self-conjugate circles as a given triangle 98

CORRIGENDA.

Vol. VIII.

Page 82, lines 27 and 28, *for* as to sign *read* as to even and odd.

Vol. IX.

„ 65, line 2 from bottom, *for* in solution *read* in this solution.
„ 66, „ 2, *for* xy *read* xy.
„ 80, „ 3 from bottom, *for* $BQ = x$ *read* $BQ = x_1$.
„ 101, „ 12, *insert* $=$ *before* $10 - \pi^2$.
„ 107, „ 18, *for* $\int z\,dx$ *read* $\int z\,dx$.

Vol. X.

„ 17, „ 10, *for* AB′ *read* A′B′.
„ 31, „ 10 from bottom, *insert* $-$ *after* $\sqrt{a}$.
„ 42, „ 12, *for* $a_1\frac{z^2}{1.2}$ *read* $\frac{a_2^2 z^2}{1.2}$.
„ 64, „ 30, *for* CA at K *read* BA at K.
„ 72, „ 16, *for* 2143 *read* 2134.
„ 85, „ 15, *for* APP′E *read* FPP′E.
„ 92, last line, *insert* is *before* $\frac{256}{525\pi^2}$.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

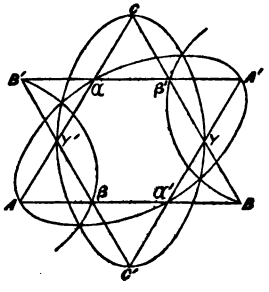
2609. (Proposed by Professor CAYLEY.)—Given three conics passing through the same four points; and on the first a point A, on the second a point B, and on the third a point C. It is required to find, on the first a point A, on the second a point B′, and on the third a point C′, such that the intersections of the lines

A B′ and AC, A′C′ and AB, lie on the first conic;
B′C′ and BA, B′A′ and BC, lie on the second conic;
C′A′ and CB, C′B′ and CA, lie on the third conic.

Solution by the PROPOSER.

Let the six intersections in question be called α, α'; β, β'; γ, γ', respectively; then BC intersects second conic in β', third in γ, CA intersects third conic in γ', first in α, AB intersects first conic in α', second in β; and we have—

A′ the intersection of $\alpha\beta'$, $\gamma\alpha$,
B′ the intersection of $\beta\gamma'$, $\alpha\beta'$,
C′ the intersection of $\gamma\alpha'$, $\beta\gamma'$;



and it has to be shown that the points A′, B′, C′ so determined lie—A′ on the first conic, B′ on the second conic, C′ on the third conic.

Taking $x = 0$, $y = 0$, $z = 0$ for the equations of the sides of the triangle ABC, the equations of the three conics may be taken to be $U = 0$, $V = 0$, $W = 0$, where the functions U, V, W are such that identically $U + V + W = 0$; and then observing that the conics pass through the points $(y = 0, z = 0)$, $(z = 0, x = 0)$, $(x = 0, y = 0)$, respectively, we see that the equations may be taken to be

$$(\ 0, -b,\ \ c, f_1, g_1, h_1)(x, y, z)^2 = 0,$$
$$(\ a,\ \ 0, -c, f_2, g_2, h_2)(x, y, z)^2 = 0,$$
$$-a,\ \ b,\ \ 0, f_3, g_3, h_3)(x, y, z)^2 = 0,$$

where $f_1+f_2+f_3 = 0$, $g_1+g_2+g_3 = 0$, $h_1+h_2+h_3 = 0$.

The coordinates of the points $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are at once found to be

$$\alpha, (\ c\ ,\ 0\ , -2g_1); \quad \alpha'\ (b\ ,\ \ 2h_1\ ,\ 0)$$
$$\beta, (-2h_2,\ a\ ,\ 0); \quad \beta'\ (0\ ,\ \ c\ ,\ 2f_2)$$
$$\gamma, (\ 0\ , -2f_3,\ b); \quad \gamma'\ (2g_3,\ \ 0\ ,\ a).$$

And hence the equations of $\beta\gamma'$, $\gamma\alpha'$, $\alpha\beta'$ are

$$\beta\gamma';\quad ax\ + 2h_2y - 2g_3z = 0,$$
$$\gamma\alpha\ ;\ -2h_1x + \ by + 2f_3z = 0,$$
$$\alpha\beta';\quad 2g_1x - 2f_1y + \ cz = 0.$$

Hence for the point A′, which is the intersection of $\gamma\alpha'$, $\alpha\beta'$, coordinates are

$$bc+4f_2f_3,\ \ 4f_3g_1+2ch_1,\ \ 4h_1f_2-2bg_1;$$

and A′ will be on the first conic if only

$$(0, -b, c, f_1, g_1,\ h_1)\ (bc+4f_2f_3,\ \ 4f_3g_1+2ch_1,\ \ 4h_1f_2-2bg_1)^2 = 0$$

viz., this equation is

$$\begin{aligned}
&-\ b\ (\ \ 16f_3^2g_1^2\ \ \ +16f_3g_1h_1c+4h_1^2c^2)\\
&+\ c\ (\ \ 16h_1^2f_2^2\ \ \ -16f_2g_1h_1b+4g_1^2b^2)\\
&+2f_1\ (\ \ 16g_1h_1f_2f_3-8g_1^2f_3b\ \ \ +8h_1^2f_2c-4g_1h_1bc)\\
&+2g_1\ (+16h_1f_2^2f_3\ \ -8g_1f_2f_3b+4h_1f_2bc-2g_1b\,.c)\\
&+2h_1\ (+16g_1f_2f_3^2\ \ +8h_1f_2f_3c+4g_1f_3bc+2h_1bc^2) = 0\,,
\end{aligned}$$

viz., this is easily found to be

$$8\,(2g_1f_3+ch_1)\,(2h_1f_2-bg_1)\,(f_1+f_2+f_3) = 0,$$

which is satisfied in virtue of $f_1+f_2+f_3 = 0$; that is, A′ is on the first conic; and similarly, in virtue of $g_1+g_2+g_3 = 0$, B′ is on the second conic; and in virtue of $h_1+h_2+h_3 = 0$, C′ is on the second conic. But the same thing appears at once by the remark that the equations of the three conics are

$$\begin{aligned}
-y\,(-2h_1x+\ \ by+2f_3z)+z\,(\ \ 2g_1x-2f_2y+\ \ cz) &= 0,\\
-z\,(\ \ 2g_1x-2f_2y+\ \ cz)+x\,(\ \ \ ax+2h_2y-2g_3z) &= 0,\\
-x\,(\ \ \ ax+2h_2y-2g_3z)+y\,(-2h_1x+\ \ by+2f_3z) &= 0.
\end{aligned}$$

It may be added that, taking (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), $(x_4, y_4\ z_4)$ as the coordinates of the four points of intersection of the three conics, the first conic is given by means of these four points and the fifth point $(y = 0,\ z = 0)$; and similarly for the other two conics; whence, denoting the determinants formed with any four columns out of the matrix

$$\left\|\begin{array}{cccccc}
x_1^2, & y_1^2, & z_1^2, & y_1z_1, & z_1x_1, & x_1y_1\\
x_2^2, & y_2^2, & z_2^2, & y_2z_2, & z_2x_2, & x_2y_2\\
x_3^2, & y_3^2, & z_3^2, & y_3z_3, & z_3x_3, & x_3y_3\\
x_4^2, & y_4^2, & z_4^2, & y_4z_4, & z_4x_4, & x_4y_4
\end{array}\right\|$$

by 1234, 1235, &c., we easily find the equations of the three conics, viz., these may be written

$$\begin{array}{rcccccc} & x^2 & y^2 & z^2 & yz & zx & xy \\ 1456\,(& 0 & +3456, & -2456, & +2356, & +2364, & +2345) = 0, \\ 2356\,(& -3456, & 0 & +1456, & +3156, & +3164, & +3145) = 0, \\ 3456\,(& 2456, & -1456, & 0 & +1256, & +1264, & +1245) = 0, \end{array}$$

the exterior factors 1456, 2356, 3456 being introduced in order to bring the equations into the above mentioned form, wherein the sum of the quadric functions is = 0.

2661. (Proposed by Dr. SALMON, F.R.S.)—Given five conics, it is of course possible in an infinity of ways to determine five constants $a, b, c,$ &c., so that

$$aU_1 + bU_2 + cU_3 + dU_4 + eU_5$$

may be either a perfect square L^2 or the product of two factors MN. Prove that the lines L all touch a conic, and that the lines M, N are conjugate with regard to that conic; from which it follows that, if M be given, N passes through a fixed point.

Solution by the PROPOSER.

Given a pair of conics U and V, it is known that there is an invariant Θ of the system, of the first degree in the coefficients of U and of the second in those of V; and that if the coefficients of U be $a, b,$ &c., and if the reciprocal coefficients of V be A, B, &c., Θ may be written

$$Aa + Bb + Cc + 2Ff + 2Gg + 2Hh.$$

If we are given five conics U, we have conditions enough to determine linearly A, B, &c., so that Θ may vanish for V and each of the five conics; and since this invariant contains the U coefficients in the first degree, a single conic V can be thus determined whose invariant Θ will vanish for every conic of the system

$$l_1V_1 + l_2U_2 + l_3U_3 + l_4V_4 + l_5U_5.$$

If this last be a perfect square L^2, the vanishing of Θ is the condition that L should touch V; or if it break up into factors MN, the vanishing of Θ is the condition that M, N should be conjugate with respect to V.

2519. (Proposed by C. W. MERRIFIELD, F.R.S.)—Given the six distances between four points in a plane; find the condition that one of them should be within the triangle formed by the other three.

Solution by the PROPOSER.

Let the triangles formed by the points taken three and three be A, B, C, D. Then, evidently, the criterion is, that the lower sign should be taken in the necessary condition $(A + B)^2 = (C \pm D)^2$, that is to say, that

$$(A^2 + B^2 + C^2 + D^2)^2 - 2(A^4 + B^4 + C^4 + D^4)$$

should be negative.

Next, let the sides (opposite) be lp, mq, nr, so that the four triangles may consist of the sets lmn, lqr, mpr, npq.

Then if we substitute $\frac{1}{16}(2l^2m^2+2m^2n^2+2n^2l^2-n^4-m^4-l^4)$ for A^2, &c., we get, as the condition that one point should lie within the triangle formed by the three others, that the following quantity should be negative:—

$$\left.\begin{aligned}
&2\Sigma\,(l^2m^2n^2p^2)+(l^4p^4+m^4q^4+n^4r^4)\\
&+l^4\left\{m^2q^2+n^2r^2-p^2\,(m^2+q^2+n^2+r^2)-m^2n^2-q^2r^2\right\}\\
&+p^4\left\{m^2q^2+n^2r^2-l^2\,(m^2+q^2+n^2+r^2)-m^2r^2-n^2q^2\right\}\\
&+m^4\left\{l^2p^2+n^2r^2-q^2\,(l^2+p^2+n^2+r^2)-l^2n^2-p^2r^2\right\}\\
&+q^4\left\{l^2p^2+n^2r^2-m^2\,(l^2+p^2+n^2+r^2)-l^2r^2-n^2p^2\right\}\\
&+n^4\left\{l^2p^2+m^2q^2-r^2\,(l^2+p^2+m^2+q^2)-l^2m^2-p^2q^2\right\}\\
&+r^4\left\{l^2p^2+m^2q^2-n^2\,(l^2+p^2+m^2+q^2)-l^2q^2-m^2p^2\right\}
\end{aligned}\right\}\quad\ldots\ldots(1).$$

The law of this is easily seen. Its form may be modified by means of the relation between the distances of four points in a plane, given by Carnot, viz.,

$$\left.\begin{aligned}
&l^2m^2n^2+l^2q^2r^2+m^2p^2r^2+n^2p^2q^2\\
&+l^2p^2\,(l^2+p^2-m^2-n^2-q^2-r^2)\\
&+m^2q^2(m^2+q^2-l^2-n^2-p^2-r^2)\\
&+n^2r^2\,(n^2+r^2-l^2-m^2-p^2-q^2)=0
\end{aligned}\right\}\quad\ldots\ldots(2),$$

which simply expresses that the tetrahedron degenerates into a plane.

The consideration that the form (1) must vanish whenever any one of the four triangles A, B, C, D vanishes leads to the inference that the form is reducible to a function of the continued product of sixteen factors,

$$(\pm l\pm m\pm n)(\pm l\pm q\pm r)(\pm m\pm p\pm r)(\pm n\pm p\pm q)\quad\ldots(3),$$

that is to say, of one of the (hemihedral) square roots of the (holohedral) expression of thirty-two factors. In fact the equation

$$(A^2+B^2+C^2+D^2)^2-2\,(A^4+B^4+C^4+D^4)=\mp 8ABCD\quad\ldots\ldots(4),$$

which is a necessary consequence of $(A+B)^2=(C\pm D)^2$, and in which the value of $16A^2$ may be written as

$$-(l+m+n)\,(l-m-n)\,(-l+m-n)\,(-l-m+n),$$

(and so for the others), shows that (1) may be reduced to the square root of (3) by the help of (2), if not without it. This remark may have some algebraical value. Meanwhile it is to be observed that it is the right-hand side only of equation (4) which yields the criterion.

2641. (Proposed by the Rev. W. ROBERTS, M.A.)—Find the value of the double integral $\iint \frac{dx\,dy}{\left\{(x^2+y^2+a^2-b^2)^2-4\,(a^2-b^2)\,x^2\right\}^{\frac{1}{2}}}$, subject to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqq 1$.

Solution by M. W. CROFTON, F.R.S.

Using elliptic coordinates μ, ν [the semi-transverse axes of the ellipse and hyperbola confocal with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which pass through the point (x, y)] and putting $c^2=a^2-b^2$, we shall have $x^2+y^2+c^2=\mu^2+\nu^2$, and $cx=\mu\nu$; so that the proposed integral becomes

$$\iint \frac{dx\,dy}{\mu^2-\nu^2}.$$

Now the element of the area is, in elliptic coordinates,

$$dS=\frac{\mu^2-\nu^2}{\sqrt{(\mu^2-c^2)(c^2-\nu^2)}}\,d\mu\,d\nu.$$

Hence the proposed integral, extended over a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, becomes

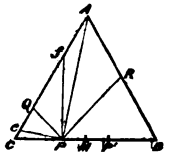
$$\int_c^a\int_0^c \frac{d\mu\,d\nu}{\sqrt{(\mu^2-c^2)(c^2-\nu^2)}}=\frac{\pi}{4}\log\frac{a+b}{a-b},$$

and the required value is $\pi\log\frac{a+b}{a-b}$.

2561. (Proposed by Captain CLARKE, R.E., F.R.S.)—1. A point is taken at random on each side of an equilateral triangle; show that the chance that these points form an acute-angled triangle is $1-\frac{3}{2\pi}\log(\frac{81}{4}e)$.

2. Find also the chance that a circle passing through the three points cuts each side of the triangle (not produced) in a second point.

Solution by the PROPOSER.



1. First, to determine the chance that the angle at the point P on BC is an obtuse angle. Let P be between C and M, the middle point of CB. Join PA, draw P*e* perpendicular to PA, meeting AC in *e*; also draw P*f* perpendicular to BC; also, Q being any point in *ef*, join PQ, and draw PR perpendicular to it, meeting AB in R. It is then clear that the points P, Q, and any point in RB, make a triangle obtuse-angled at P. Any point in C*e*, with any point in AB, also subtend an obtuse angle at P, but no point in *f*A will do so. Let $CQ=y$, $CP=x$; then, the angle CPQ being the complement of RPB, it is easy to shew that, if $CB=1$, then

$$RB=\frac{(1-x)(2x-y)}{x+y};$$

and supposing R to coincide with A, we derive for C*e* the value

$$Ce=y_1=2x+3-\frac{6}{2-x},\quad \text{also}\quad Cf=y_2=2x.$$

The total number of ways, then, in which points on AB, AC can be combined with P so as to form a triangle obtuse-angled at P is

$$2\int_0^{\frac{1}{2}}\left\{y_1+\int(\mathrm{RB})\,dy\right\}dx = 2\int_0^{\frac{1}{2}}\left(y_1+\int_{y_1}^{y_2}\left\{\frac{3x(1-x)}{x+y}-1+x\right\}dy\right)dx$$

$$= 2\int^{\frac{1}{2}}\left\{y_1+3x(1-x)\log\frac{x+y_2}{x+y_1}-(1-x)(y_2-y_1)\right\}dx$$

$$= 2\int_0^{\frac{1}{2}}\left\{3x(1-x)\log\frac{2-x}{1-x}-x\right\}dx = \tfrac{3}{8}+\tfrac{3}{4}\log\tfrac{3}{4},$$

which is the chance that the triangle formed by the three random points is obtuse-angled at P. The chance, then, that *some* of the angles are obtuse is

$$1-\tfrac{3}{4}\log(\tfrac{3}{64}\epsilon).$$

It is interesting, in connection with this, to remark that, P, Q, R being random points on the sides, the chance that the triangle formed by AP, BQ, CR is acute-angled is $1-3\log(\frac{3}{64}\epsilon)$. For we may show that, if QB and CR intersect on the semicircle whose diameter is CB, the relation between $AQ=u$ and $AR=v$ is $1-2(u+v)+uv=0$. Supposing, then, the point of intersection O to be without the semicircle, the triangle is obtuse-angled at O; and the number of ways in which PQR can be taken to produce this result is

$$\int_0^{\frac{1}{2}} v\,du = \int_0^{\frac{1}{2}}\left(2-\frac{3}{2-u}\right)du = 1+3\log\tfrac{1}{2} = \log(\tfrac{3}{64}\epsilon),$$

consequently the number of ways in which the points can be taken so as to produce an acute-angled triangle is $1-3\log(\frac{3}{64}\epsilon)$.

2. We shall first determine the number of circles which do not cut AB and AC twice. Let R, Q be any two points on these sides, and let the circle passing through A, R, Q cut BC in P, P′: also let $CQ=y$, $BR=z$, $CP=x$, $CP'=x'$, then we have

$$xx'=y,\quad (1-x)(1-x')=z,\quad x+x'=1+y-z;$$

therefore

$$x'-x=\left\{(1+y-z)^2-4y\right\}^{\frac{1}{2}},$$

which is the value of PP′. When this vanishes, $z=(1-y^{\frac{1}{2}})^2$. Now QR and any point between P and P′ determine a circle which encloses A, that is, does not again cut AB and AC. The total number of circles therefore fulfilling this condition is

$$\int_0^1\int_0^{(1-y^{\frac{1}{2}})^2}\left\{(1+y-z)^2-4y\right\}^{\frac{1}{2}}dy\,dz = \int_0^1\left\{y\log y+\tfrac{1}{2}(1-y)^2\right\}dy = \tfrac{1}{12}.$$

Therefore the number of circles which enclose either A, B, or C is $\frac{1}{4}$; or the chance that the circle cuts each side in a second point is $\frac{3}{4}$.

2620. (Proposed by G. M. Smith, B.A.)—1. A rigid body is rotating round a fixed axis; suddenly this axis becomes free, and a parallel axis becomes fixed elsewhere in the body. Find the new initial angular velocity, and the locus of the new axis if the angular velocity is unchanged.

2. A right prism, whose base is a regular polygon of n sides, falls (without sliding) down a rough inclined plane, the contact being supposed to take place only at the corners; find its angular velocity just before the mth impact.

3. Hence deduce the law of motion of a cylinder or sphere down a rough inclined plane.

Solution by the Rev. R. TOWNSEND, F.R.S.; *the* PROPOSER; *and others.*

1. There being no change of *vis viva* under the circumstances supposed, we have $\omega^2 I = \omega'^2 I$, ω and ω' being the angular velocities, and I and I′ the moments of inertia round the new and old axes. If $\omega' = \omega$, then $I' = I$, and the locus in question is consequently a cylinder of revolution round the parallel axis passing through the centre of gravity of the body.

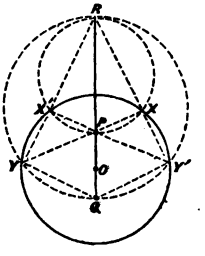
2. If M be the mass of the prism, I its moment of inertia round any one of its edges, a the length of any side of the polygon, and i the inclination of the plane to the horizon; then, the *vis viva* of the body being due entirely to the action of gravity, we have, just before the mth impact $\omega^2 I = 2Mg.ma \sin i$, the centre of gravity of the body having been lowered through the height $ma \sin i$, since the commencement of the motion.

3. If r be the radius of the cylinder or sphere, and v the velocity of its centre of gravity after rolling from rest through the space s on the plane; since then I, for the former $= \frac{1}{2}Mr^2$, and for the latter $= \frac{2}{3}Mr^2$, and since v^2 for both $= \omega^2 r^2$, therefore v^2 for both $= 2fs$, or the motion for both is uniform acceleration; f being $= \frac{2}{3}g \sin i$ for the former, and $= \frac{5}{7}g \sin i$ for the latter.

2626. (Proposed by J. J. WALKER, M.A.)—To draw through a given point within a circle a chord which shall be divided at the point in extreme and mean ratio.

Solution by the Rev. R. TOWNSEND, F.R.S.; *the* PROPOSER; *and others.*

The following construction will place the two chords, when real, in position, whatever be the ratio of their section, viz., if O be the centre of the given circle, P the given point, Q the point on the diameter passing through it for which OQ = OP, and R the point of external section of QP in the given ratio; then the two circles on PR and QR as diameters will evidently intersect the circle at the two pairs of extremities X and Y, X′ and Y′ of the two chords required.



This is evident from the parallelism of the two pairs of lines PX and QY′, PX′ and QY, and from the equality of the two pairs PY and QY′, PY′ and QY.

2532. (Proposed by W. S. B, WOOLHOUSE, F.R.S.)—If within a given enclosed area three points be taken at random as the apices of a triangle, the average of the square of the area of all such triangles will be reduced to one-third the value if one of the points be fixed at the centre. Also if within a given volume of space four points be taken at random as the apices of a tetrahedron, the average of the square of the volume of all such tetrahedra will be reduced to one-fourth the value if one point be fixed at the centre. Show that this theorem is true when the enclosed area or volume of space is *of any form whatever*.

2514. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Let a given volume of space, having a convex boundary *of any form whatever*, be referred to its centre of gravity and the principal axes of rotation, and let M denote the total volume and h, k, l what may be called compressed radii of gyration round the axes, so that

$$\mathrm{M}.h^2 = \iiint x^2\,dx\,dy\,dz, \quad \mathrm{M}.k^2 = \iiint y^2\,dx\,dy\,dz, \quad \mathrm{M}.l^2 = \iiint z^2\,dx\,dy\,dz;$$

then

1. If *four* points be taken at random within the given volume, the average square of the volume of the tetrahedron which connects them is $\frac{1}{5}h^2k^2l^2$.

2. If *six* points be taken at random, within the given volume, as the corners of a hexaedron, the probability that two of the points shall be re-entrant is $\dfrac{10h^2k^2l^2}{\mathrm{M}^2}$.

2556. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Four solids, *having any forms whatever*, being given, show how to determine sets of six points fixed in each, such that if the four solids be placed in any positions in space and four points be taken arbitrarily, one in each solid, as the apices of a tetrahedron, the average square of volume in respect of all such tetrahedra, shall always be equal to that of the 1296 tetrahedra formed by connecting all the groups obtainable from the four invariable sets of six points.

Solution by the PROPOSER.

To begin with the first case of Quest. 2532, suppose (x_1, y_1), (x_2, y_2), (x_3, y_3) to be the coordinates of the three points referred to rectangular axes, let M denote the magnitude of the given enclosed area; dm_1, dm_2, dm_3 differential elements of the area in the immediate vicinity of the respective points; and Δ the area of the variable triangle. Then

$$2\Delta = x_3(y_1 - y_2) - y_3(x_1 - x_2) + x_1y_2 - x_2y_1,$$

$$\therefore\ 4\Delta^2 = x_3^2(y_1-y_2)^2 + y_3^2(x_1-x_2)^2 + (x_1y_2 - x_2y_1)^2 + \mathrm{P}x_3 + \mathrm{Q}y_3 + \mathrm{R}x_3y_3,$$

where P, Q, R are functions of (x_1, y_1), (x_2, y_2), and independent of (x_3, y_3).

Now the average value of Δ^2 is $= \iiint \frac{dm_1}{\mathrm{M}}\frac{dm_2}{\mathrm{M}}\frac{dm_3}{\mathrm{M}}\Delta^2$; and if the axes of coordinates be the principal axes through the centre of gravity of the given area, and h, k denote the radii of gyration of the area round those axes, we shall have generally

$$\int \frac{dm}{\mathrm{M}}x^2 = h^2, \ \int \frac{dm}{\mathrm{M}}y^2 = k^2; \ \int \frac{dm}{\mathrm{M}}xy = 0, \ \int \frac{dm}{\mathrm{M}}x = 0, \ \int \frac{dm}{\mathrm{M}}y = 0.$$

Hence we find

$$4\int \frac{dm_3}{\mathrm{M}}\Delta^2 - h^2(y_1^2 - 2y_1y_2 + y_2^2) + k^2(x_1^2 - 2x_1x_2 + x_2^2) + (x_1^2y_2^2 - 2x_1y_1x_2y_2 + x_2^2y_1^2).$$

Similarly, with the next integration,

$$4\iint \frac{dm_2}{M}\frac{dm_3}{M}\Delta^2 = h^2(y_1^{2*}+h^2)+k^2(x_1^{2*}+h^2)+(k^2x_1^{2*}+h^2y_1^2),$$

or
$$\iint \frac{dm_2}{M}\frac{dm_3}{M}\Delta^2 = \frac{h^2k^2}{2}\left(\frac{x_1^2}{h^2}+\frac{y_1^2}{k^2}+1\right) \ldots\ldots\ldots\ldots (\alpha);$$

therefore
$$\iiint \frac{dm_1}{M}\frac{dm_2}{M}\frac{dm_3}{M} = \frac{h^2k^2}{2}(1+1+1) = \tfrac{3}{2}h^2k^2 \quad \ldots\ldots (\beta),$$

which is the average value of the square of the area of the triangle when the three points are taken arbitrarily within the given surface. The expression (α) exhibits the same when one of the points (x_1, y_1) is fixed, and this value is unchanged when the point is supposed to be taken anywhere in the periphery of an ellipse concentric with the given area. When the point (x_1, y_1) is fixed at the centre, or origin of coordinates, the value becomes $\frac{1}{2}h^2k^2$, which is one-third of the general average, as was required to be shown.

In the case of three dimensions, if V denote the volume of the variable tetrahedron,

$$3V = z_1 . \Delta(2, 3, 4) - z_2 . \Delta(3, 4, 1) + z_3 . \Delta(4, 1, 2) - z_4 . \Delta(1, 2, 3),$$

wherein $\Delta(2, 3, 4)$, $\Delta(3, 4, 1)$, &c. denote the areas of triangles determined by orthogonal projection of the several points on the plane of xy. This formula was established in one of my early papers "On the Computation of Superficies and Solids," contained in Leybourne's *Mathematical Repository*, Vol. VI., and equivalent expressions may now be found in most works on Analytical Geometry. The algebraic sign of the area thus determined is dependent upon the order in which the four points are taken; but as we are about to deal with the square of the area, this result is always positive, and is determined by the formula

$$9V^2 = z_1^2(\Delta 234)^2 + z_2^2(\Delta 341)^2 + z_3^2(\Delta 412)^2 + z_4^2(\Delta 123)^2 + \mathrm{P}z_1z_2 + \&c.$$

The axes of coordinates being, as before, the principal central axes, we shall, in integrating with respect to any variable point, have

$$\int \frac{dm}{M}x^2 = h^2, \quad \int \frac{dm}{M}y^2 = k^2, \quad \int \frac{dm}{M}z^2 = l^2 \quad \ldots\ldots\ldots (1),$$

$$\int \frac{dm}{M}yz = 0, \quad \int \frac{dm}{M}zx = 0, \quad \int \frac{dm}{M}xy = 0 \ldots\ldots\ldots (2),$$

$$\int \frac{dm}{M}x = 0, \quad \int \frac{dm}{M}y = 0, \quad \int \frac{dm}{M}z = 0 \quad \ldots\ldots\ldots (3).$$

In operating upon the expression just given for $9V^2$, the six terms $\mathrm{P}z_1z_2 + \&c.$, involving binary products of $z_1z_2z_3z_4$, must by virtue of (3) all disappear after any three integrations.

By (β), $\iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M} z_1^2(\Delta 234)^2 = \frac{3}{2}h^2k^2z_1^2.$

By (α), $\left\{\begin{array}{l} \iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M} z_2^2(\Delta 341)^2 \\ \iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M} z_3^2(\Delta 412)^2 \\ \iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M} z_4^2(\Delta 123)^2 \end{array}\right\}$ each $= \frac{h^2k^2}{2}\left(\frac{x_1^2}{h^2}+\frac{y_1^2}{k^2}+1\right).l^2,$

and, as has been remarked, all subsequent terms = 0. Hence we get

$$9\iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M}V^2 = \tfrac{1}{2}h^2k^2z_1^2 + \tfrac{1}{2}h^2k^2l^2\left(\frac{x_1^2}{h^2}+\frac{y_1^2}{k^2}+1\right),$$

$$\therefore \quad \iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M} = \frac{h^2k^2l^2}{6}\left(\frac{x_1^2}{h^2}+\frac{y_1^2}{k^2}+\frac{z_1^2}{l^2}+1\right) \ldots\ldots\ldots(\gamma),$$

$$\iiiint \frac{dm_1}{M}\frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M} = \frac{h^2k^2l^2}{6}(1+1+1+1) = \tfrac{2}{3}h^2k^2l^2 \ldots.(\delta),$$

which is, therefore, the average value of the square of the volume of a tetrahedron connecting four arbitrary points. The formula (γ) expresses the value when one of the points (x_1, y_1, z_1) is fixed, and it has a constant value when that point is taken in the surface of a determinate concentric ellipsoid. When the immoveable point (x_1, y_1, z_1) is at the centre, the value is $\frac{1}{6}h^2k^2l^2$ and is one-fourth of the general average (δ).

The reasoning obviously applies to all boundaries, whether continuous or not, regular or irregular.

The formula (δ) establishes the first part of Quest. 2514; and the second part of the same question is readily deduced from it as follows:—If v denote the volume of the tetrahedron formed by connecting four of the six points, the separate and independent probabilities that the fifth and sixth points shall fall within v are each $\frac{v}{M}$, and the joint probability that they shall both fall within v is $\frac{v^2}{M^2}$. Hence, when the several points take all positions within the given volume of space, the probability that a specified pair of points shall be both reentrant is $\frac{2}{3}\cdot\frac{h^2k^2l^2}{M^2}$; and as the same identical amount of probability attaches to each of the fifteen pairs of the six points, it follows that the complete probability of one or other of the several pairs being reentrant is

$$\frac{2}{3}\cdot\frac{h^2k^2l^2}{M^2}\times 15 = \frac{10h^2k^2l^2}{M^2}.$$

The theorem stated in Quest. 2556 may be shown to be true more generally, when in place of the square of the volume of the tetrahedron we substitute any arbitrary algebraic function which involves any one set of coordinates to the second degree. The degree of this function will not be changed by altering the position of the axes of coordinates. It will be convenient if these axes be the principal axes through the centre of gravity of the volume of space in which the point $x_4y_4z_4$ is to be taken. Let the proposed function, after the coordinates are transferred to these particular axes, be

$$U = Ax^2 + By^2 + Cz^2 + ayz + bzx + cxy + \alpha x + \beta y + \gamma z + G,$$

where A, B, C, a, b, c, α, β, γ, G are functions of $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$, involving each set of coordinates to the second degree. Then the average value of U with respect to the variable point $x_4y_4z_4$ is evidently

$$(U) = \int \frac{dm_4}{M}U = Ah^2 + Bk^2 + Cl^2 + G \ldots\ldots\ldots\ldots\ldots(\epsilon).$$

But for certain special values of $x_4y_4z_4$ we have

$$\begin{aligned}
U_{h\sqrt{3},\,0,\,0} &= 3Ah^2 + ah\sqrt{3} + G,\\
U_{-h\sqrt{3}\,0,\,0} &= 3Ah^2 - ah\sqrt{3} + G,\\
U_{0,\,k\sqrt{3},\,0} &= 3Bk^2 + \beta k\sqrt{3} + G,\\
U_{0,\,-k\sqrt{3},\,0} &= 3Bk^2 - \beta k\sqrt{3} + G,\\
U_{0,0,\,l\sqrt{3}} &= 3Cl^2 + \gamma l\sqrt{3} + G,\\
U_{0,\ ,-l\sqrt{3}} &= 3Cl^2 - \gamma l\sqrt{3} + G,
\end{aligned}$$

$$\therefore (U) = \frac{1}{6}\left\{U_{h\sqrt{3},\,0,\,0} + U_{-h\sqrt{3},\,0,\,0} + U_{0,\,k\sqrt{3},\,0} + U_{0,\,-k\sqrt{3},\,0} + U_{0,\,0,\,l\sqrt{3}} + U_{0,\,0,\,-l\sqrt{3}}\right\}$$

That is, the average value of the function, when $x_4y_4z_4$ takes all the positions assigned to it, is equal to the average value when $x_4y_4z_4$ successively occupies the six determinate points $\pm h\sqrt{3}, 0, 0$; $0, \pm k\sqrt{3}, 0$; $0, 0, \pm l\sqrt{3}$. Now a concentric and coaxial rectangular parallelepiped, the semi-sides of which are $h\sqrt{3}$, $k\sqrt{3}$, $l\sqrt{3}$, has the same radii of gyration (h, k, l) as the solid in which $x_4y_4z_4$ is taken; and therefore, by (ϵ), the average value (U) would be the same if the point $x_4y_4z_4$ were taken either arbitrarily within the parallelepiped or upon each of the six determinate points before mentioned which are obviously the six centres of its six rectangular faces.

By similarly transferring the coordinates to the principal axes of the other three solids, it may be shown that in each case the average value of the function U is unchanged in magnitude if the point, instead of traversing the volume of the solid, is made to successively occupy the six centres of the faces of the dynamically equivalent parallelepiped. Hence by superposition we arrive at the stated theorem, that when the four points vary throughout the respective solids, placed in any positions, the average value of the function is the same as the average of the 1296 values deduced by associating all the groups obtainable from these invariable sets of six points.

An analogous theorem also exists with respect to certain determinate sets of *four* points in each solid. In fact it is generally true for any determinate sets of points which, regarded as material systems, have the same principal axes and radii of gyration as the respective solids. It might be shown, however, that the conditions here stated admit of being realized only when the number of points in each set is either 4, 6, 8, 12, or 20; that is, the number of corners of one or other of the five regular polyhedra.

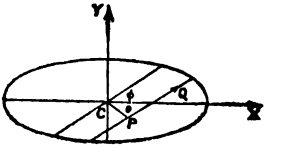
NOTE.—It will be observed that the squares of the radii of gyration of the given volume of space round the coordinate axes are respectively equal to $k^2 + l^2$, $l^2 + h^2$, $h^2 + k^2$. The symbols h, k, l are what in Quest. 2514 I have designated as compressed radii of gyration, and, in fact, represent what the several radii would become if the total volume of the body were compressed into one of the three coordinate planes.

Solution of Question 2532, *by the* Rev. J. WOLSTENHOLME, M.A.

Let C be the centre of gravity of any lamina; CX, CY the principal axes of inertia in the plane; and let the area be unity. Then the average value of the square of the triangle PQR is

$$\tfrac{1}{4}\iiint PQ^2 \cdot y^2 \cdot dp\, dq\, dr,$$

y being the distance of R from PQ, and dp, dq, dr elementary areas about PQR.



But $\int y^2\,dr$ = moment of inertia about the line PQ
$= a^2\cos^2\phi + b^2\sin^2\phi + \mathrm{CP}^2\sin^2(\theta+\phi)$;

a, b being radii of gyration about CX, CY; and θ, ϕ the angles which CP and PQ make with CX; therefore the average value

$$= \tfrac{1}{\pi}\iint r^2\left\{a^2\cos^2\phi + b^2\sin^2\phi + \mathrm{CP}^2\sin^2(\theta+\phi)\right\}dp\,dq,\ (\mathrm{PQ}\equiv r).$$

But $\int r^2\cos^2\phi\,dq$ = moment of inertia about a st. line through P parallel to CY
$= b^2 + \mathrm{CP}^2\cos^2\theta$.

So $\int r^2\sin^2\phi\,dq = a^2 + \mathrm{CP}^2\sin^2\theta$, and $\int r^2\sin^2(\theta+\phi)\,dq = a^2\cos^2\theta + b^2\sin^2\theta$,

or average value $= \tfrac{1}{\pi}\int(2a^2b^2 + 2a^2\rho^2\cos^2\theta + 2b^2\rho^2\sin^2\theta)\,dp$, $(\mathrm{CP}\equiv\rho)$;

and $\int \rho^2\cos^2\theta\,dp$ = moment of inertia about CY $= b^2$; so $\int \rho^2\sin^2\theta\,dp = a^2$;

therefore, finally, the average value of the square of the triangle $= \tfrac{1}{4}a^2b^2$.

But average value of square of the triangle CPQ $= \tfrac{1}{\pi}\iint \mathrm{CP}^2y'^2\,dp\,dq$ (y' being distance of Q from CP),

and $\int y'^2\,dq$ = moment of inertia about CP $= a^2\cos^2\theta + b^2\sin^2\theta$,

therefore the average value $= \tfrac{1}{\pi}\int(a^2\rho^2\cos^2\theta + b^2\rho^2\sin^2\theta)\,d\theta$
$= \tfrac{1}{2}a^2b^2$ (for the same reason as before).

The investigation for three dimensions is of much the same kind, referring to the principal axes of inertia at the centre of gravity; and the corresponding results are $\tfrac{8}{3}a^2b^2c^2$ and $\tfrac{1}{3}a^2b^2c^2$; b^2+c^2, c^2+a^2, a^2+b^2 being the squares of the radii of gyration about the principal axes.

2348. (Proposed by T. COTTERILL, M.A.)—In the system of conics touching two lines at fixed points, the focal cubic cuts a conic in points such that its normals at these points pass through the point of intersection of the two lines.

Solution by J. DALE; H. TOMLINSON; *and others.*

Referring the conics to the triangle formed by the two lines and the chord of contact, the equation to any one will be of the form $ux^2 + 2u'yz = 0$, and the coordinates (x, y, z) of a focus will be given by the equations

$$u'(by+cz)^2 + 2ubcx^2 + k = 0,$$
$$u'a^2y^2 - 2ucy(cz+ax) + k = 0,$$
$$u'a^2z^2 - 2ubz(ax+by) + k = 0,$$

the coordinates being trilinear. Eliminating u', u, k; dividing by $ax+by+cz$, and reducing, we get for the equation to the locus of the foci

$$x(y^2-z^2) + 2(y\cos\mathrm{B} - z\cos\mathrm{C})\,yz = 0.$$

This cubic meets any conic of the system $ux^2 + 2u'yz = 0$ in the points where $x = 0$ meets the conic; that is, in the points of contact B, C; and also in the points where the conic meets the curve

$$u'(y^2-z^2) + ux(z\cos\mathrm{C} - y\cos\mathrm{B}) = 0.$$

But this curve is an equilateral hyperbola, passing through A ($y=0$, $z=0$), and through the centre of $ux^2+2u'yz=0$, and having its axes parallel to the axes of the conic; therefore the normals at the points of intersection pass through A.

Cor.—The focal cubic has a double point at A, the tangents being $y-z=0$ and $y+z=0$, and cuts BC in the foot of the perpendicular from A. Every hyperbola of the system cuts BC in the points ($x=0$, $y-z=0$) and ($x=0$, $y+z=0$).

2364. (Proposed by S. Watson.)—Through each pair of the angles of a triangle circles are described so that the internal segments (in reference to the triangle) are similar. Find the locus of the intersection of the radical axes of the three circles.

Solution by J. Dale; *the* Proposer; *and others.*

Let each of the segments subtend an angle 2θ at the corresponding centre; then, the given triangle being taken as that of reference, the trilinear coordinates (x, y, z) of the centres O_1, O_2, O_3 are respectively

$$\{-\cos\theta, \cos(C-\theta), \cos(B-\theta)\},\ \{\cos(C-\theta), -\cos\theta, \cos(A-\theta)\},$$
$$\{\cos(B-\theta),\ \cos(A-\theta),\ -\cos\theta\}.$$

The radical axes are the straight lines through A, B, C, perpendicular to O_2O_3, O_3O_1, O_1O_2, respectively. The equation to O_2O_3 is

$$\{\cos^2\theta-\cos^2(A-\theta)\}x+\{\cos(A-\theta)\cos(B-\theta)+\cos(C-\theta)\cos\theta\}y$$
$$+\{\cos(B-\theta).....\}z=0;$$

the equation to the perpendicular to O_2O_3 is $y\sin(B+\theta)-z\sin(C+\theta)=0$;
the equation to the perpendicular to O_3O_1 is $z\sin(C+\theta)-x\sin(A+\theta)=0$;
the equation to the perpendicular to O_1O_2 is $x\sin(A+\theta)-y\sin(B+\theta)=0$.

Eliminating θ, we find for the equation to the locus

$$yz\sin(B-C)+zx\sin(C-A)+xy\sin(A-B)=0,$$

which represents an equilateral hyperbola, circumscribing the triangle and passing through its centre of gravity, since the equation is satisfied by

$$x\sin A=y\sin B=z\sin C.$$

The centre (ξ, η, ζ) of the hyperbola is given by the equations

$$\frac{\xi}{\sin A\sin^2(B-C)}=\frac{\eta}{\sin B\sin^2(C-A)}=\frac{\zeta}{\sin C\sin^2(A-B)},$$

therefore
$$(a\xi)^{\frac{1}{2}}+(b\eta)^{\frac{1}{2}}+(c\zeta)^{\frac{1}{2}}=0,$$

that is, the centre lies on the maximum inscribed ellipse. The straight lines AO_1, BO_2, CO_3 intersect in the point $x\cos(A-\theta)=y\cos(B-\theta)=z\cos(C-\theta)$, and this point also lies on the hyperbola [a property which was given by the Editor as Quest. 1281, in the *Educational Times* for July, 1862].

If O_4, O_5, O_6 are the centres of circles whose external segments are equal to the above internal segments, then the radical axes of these circles intersect in the point $x\cos(A+\theta) = y\cos(B+\theta) = z\cos(C+\theta)$, which lies on the hyperbola, and the lines AO_4, BO_5, CO_6 intersect in the point $x\sin(A-\theta) = y\sin(B-\theta) = z\sin(C-\theta)$, which also lies on the hyperbola.

The four points $x\sin(A+\theta)=$&c., $x\cos(A+\theta)=$&c., $x\cos(A-\theta)=$&c., $x\sin(A-\theta)=$&c., form a quadrilateral whose diagonals intersect in the points

$$\frac{x}{\sin A} = \frac{y}{\sin B} = \frac{z}{\sin C}, \quad \frac{x}{\cos A} = \frac{y}{\cos B} = \frac{z}{\cos C};$$

$$\frac{x}{\sin(B-C)} = \frac{y}{\sin(C-A)} = \frac{z}{\sin(A-B)}.$$

2430. (Proposed by the Rev. R. H. Wright, M.A.)—A cycloid revolving about its axis generates a solid which moves in the direction of the axis through a fluid with a given velocity; compare the resistances in the two cases when the vertex and the base are respectively foremost.

Solution by J. Dale; J. J. Walker, M.A.; E. McCormick; *and others.*

Let ρ = density of fluid, v = velocity; then, when the vertex is first,

$$\text{whole resistance } (R_1) = \pi\rho v^2 \int y \left(\frac{dy}{ds}\right)^2 dy.$$

By the nature of the cycloid $x = a(1-\cos\theta)$, $y = a(\theta+\sin\theta)$;

therefore $R_1 = 2\pi\rho v^2 a^2 \int_0^{\pi} (\theta+\sin\theta)\cos^4\tfrac{1}{2}\theta\, d\theta = \tfrac{1}{24}\pi\rho v^2 a^2 (9\pi^2-16)$.

When the base is foremost, resistance $(R_2) = \frac{1}{2}\pi^2\rho v^2 a^2$;

therefore $R_1 : R_2 = 9\pi^2-16 : 12\pi^2$.

2548. (Proposed by J. J Walker, M.A.)—Given $\cos^2 x\cos^2 a = \sin^2 a\cos^2 a - 4\cos x\sin(x-a)\sin^2 a$; it is required to find the angle x.

Solution by the Proposer.

Put $x = \theta + a$, and develope; then

$$\cos^2\theta\cos^4 a - 2\sin\theta\cos\theta\sin a\cos^2 a = \cos^2\theta\sin^2 a\cos^2 a - 4\sin\theta\cos\theta\sin^3 a\cos a + 4\sin^2\theta\sin^4 a,$$

or $\cos\theta\,(\cos\theta\cos\alpha - 2\sin\theta\sin\alpha)\cos^2\alpha = (\cos\theta\cos\alpha - 2\sin\theta\sin\alpha)^2\sin^2\alpha$,
whence either $\cos\theta\cos\alpha - 2\sin\theta\sin\alpha = 0$ .....(1),
or else $\cos\theta\cos^2\alpha = (\cos\theta\cos\alpha - 2\sin\theta\sin\alpha)\sin^2\alpha$ ....(2).
From (1), $\cos(\theta+\alpha) = \sin\theta\sin\alpha$, or $\cos x = \sin(x-\alpha)\sin\alpha$,

whence $$\tan x = \frac{1+\sin^2\alpha}{\sin\alpha\cos\alpha} = 2\tan\alpha + \cot\alpha.$$

From (2), $\cos^2\alpha\cos(\theta+\alpha) = \sin^2\alpha\cos(\theta+\alpha) - \sin\theta\sin\alpha$,
or $\cos x\,(\sin^2\alpha - \cos^2\alpha) = \sin(x-\alpha)\sin\alpha = \sin x\sin\alpha\cos\alpha - \cos x\sin^2\alpha$,

whence $$\tan x = \frac{2\sin^2\alpha - \cos^2\alpha}{\sin\alpha\cos\alpha} = 2\tan\alpha - \cot\alpha.$$

[The equation may be otherwise solved as follows :—
$$\cos^2\alpha\,(\cos^2 x - \sin^2\alpha) + 4\cos x\sin(x-\alpha)\sin^2\alpha = 0,$$
or $$\cos^2\alpha\cos(x+\alpha)\cos(x-\alpha) + 4\cos x\sin(x-\alpha)\sin^2\alpha = 0;$$
and dividing by $\cos x\cos(x-\alpha)$ and developing, we get
$$\tan^2 x - 4\tan\alpha\tan x + 4\tan^2\alpha = \cot^2\alpha;$$
therefore $$\tan x = 2\tan\alpha \pm \cot\alpha,$$
which agrees with Mr. WALKER's results.]

2426. (Proposed by C. TAYLOR, M.A.)—If F_n denote the fraction $\frac{1+a^2+a^4+\ldots+a^{2n}}{a+a^3+\ldots+a^{2n-1}}$, prove that, in general,

(1) $\ldots F_n > \frac{n+1}{n} + \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2$, (2) $\ldots F_n > \frac{1}{2}\cdot\frac{n+1}{n}\left(a+\frac{1}{a}\right)$, (3) $\ldots \frac{n}{n+1}F_n$ increases with n.

Solution by the PROPOSER.

It may be observed that the "inductive" arrangement is not essential to the solution of Quest. 1521, given on p. 105 of Vol. VI. of the *Reprint*.

(1). It is there shown that

$$F_n - F_{n+1} < \frac{1}{n(n+1)} < \frac{1}{n} - \frac{1}{n+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (a).$$

By successive applications of (a), we have $F - F_n < 1 - \frac{1}{n}$;

therefore $$F_n > a + \frac{1}{a} - 1 + \frac{1}{n} > \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 + \frac{n+1}{n};$$

A fortiori, $$F_n > \frac{n+1}{n}.$$

(2) and (3). By (*a*) we have

$$F_n - F_{n+1} < \frac{1}{(n+1)^2} \cdot \frac{n+1}{n} \cdot \frac{1}{(n+1)^2} F_n, \text{ a fortiori.}$$

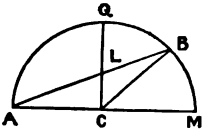
Write $n-1$ for n and transpose; then

$$F_n > \left(1-\frac{1}{n^2}\right) F_{n-1}, \quad \frac{n}{n+1} F_n > \frac{n-1}{n} F_{n-1} > \ldots > \frac{1}{2}\left(a+\frac{1}{a}\right).$$

2584. (Proposed by the late Rev. R. H. WRIGHT, M.A.) — AQM is a semicircle, QC the vertical radius, AB a chord cutting QC in L so that $QC = n \cdot CL$; find the ratio between the triangles BCL and ACL.

Solution by S. WATSON; *the* REV. J. WOLSTENHOLME, M.A.; *and many others.*

1. Draw the lines as in the question; then



$$AL^2 = CQ^2 + CL^2 = (n^2+1)\,CL^2,$$

and $$AL \cdot LB = CQ^2 - CL^2 = (n^2-1)\,CL^2;$$

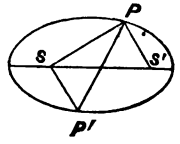
therefore $$\frac{\text{triangle ACL}}{\text{triangle BCL}} = \frac{AL}{BL} = \frac{n^2+1}{n^2-1}.$$

2. *Otherwise:* $$\frac{\Delta BCL}{\Delta ACL} = \frac{BL}{AL} = \frac{AB}{AL} - 1 = \frac{2\cos A}{\sec A} - 1 = \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A};$$

and $$\frac{CL}{CQ} = \frac{1}{n} = \frac{CL}{CA} = \tan A; \quad \therefore \frac{\Delta ACL}{\Delta BCL} = \frac{n^2+1}{n^2-1}.$$

2535. (Proposed by the Rev. W. ROBERTS, M.A.)—The ends of any diameter of an ellipse are joined with one of the foci; prove that the angles made by the joining lines with the major axis are the eccentric angles corresponding to the ends of a focal chord.

Solution by the REV. JOSEPH WOLSTENHOLME, M.A.



The angle P'SS' = PS'S, therefore the angles are equal to PSS' and PS'S, or θ and ϕ suppose; hence putting 2σ for the perimeter of the triangle SPS' and a for the side SS', we have

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{\sigma-a}{\sigma} = \frac{1-e}{1+e}, \text{ or } \cos\frac{\theta+\phi}{2} = e\cos\frac{\theta-\phi}{2},$$

which is the condition that the chord

$$\frac{x}{a}\cos\frac{\theta+\phi}{2} + \frac{y}{b}\sin\frac{\theta+\phi}{2} = \cos\frac{\theta-\phi}{2},$$

joining the points whose eccentric angles are θ and ϕ, should pass through a focus.

NOTE ON RANDOM LINES. By W. S. B. Woolhouse, F.R.A.S.

The term "random line" is in itself indeterminate as to its interpretation, requiring some sort of restrictive adjunct to impart to it a definite meaning, and this ambiguity naturally arises out of the number of variable elements upon which its position necessarily depends. In considering any problem in chances, relating to random lines, the definition of the lines so designated, or of the mathematical conditions to which they are subject, ought evidently to be elicited from the particular mode in which they are assumed to be generated. There can be no doubt whatever as to the precise signification of a random line drawn from a given point, as such line, for obvious reasons, must be regarded as taking a random direction from the given point, and therefore as equally likely to fall upon equal angular spaces round the same. And just as intelligible and distinct a meaning attaches to a random chord drawn in a given circle, since the mechanical process of its generation, by actual drawing, necessarily implies a commencement at one of its extremities, and this extremity must evidently be a random point on the circumference, the interpretation of which is well understood. In my solution to Quest. 1894 (*Reprint*, Vol. V., p. 110), I have shown at the outset that the conditions which subsist under such circumstances are exactly the same as when the chord is supposed to join two random points taken in the circumference. In each of these cases the definition arrived at is obviously the only one that is compatible with the existing conditions.

Again, the phrases—

1. A random line drawn from a random point,
2. A random line connecting two random points,

are not definite in their meaning until it is stated how, or under what limitations, the random points are supposed to be taken. If they are taken on a given finite area, the two sets of conditions are then essentially different, both in form and result, since if we conceive a random point to be taken as the origin of the line, the line itself may equally proceed in all directions round this origin in the first case, but obviously not so in the second. If, however, the random points are supposed to be taken on an infinitely extended plane, the two cases just stated will then merge into like conditions, and either of them will determine a random line just the same as that recently adopted by Mr. Crofton,* who ingeniously assumes it to be one of an assemblage of lines thus described :—"Divide the angular space round any point into a number of equal infinitesimal angles, and for every direction let the plane be ruled with an infinity of parallel equidistant lines, the common infinitesimal distance being the same for every set of parallels." The perfectly equal distribution over the plane, maintained by this assemblage of lines and their intersections, must necessarily impart an elementary and interesting character to the results of investigations relating to the intersections of random lines, when these are made to depend upon such an uniform combination of materials. The conditions conveyed by these equally distributed combinations are virtually the same as those which subsist in either of the two foregoing cases, provided that the random points which determine the lines are conceived to range absolutely over an infinite plane ; but such conditions as these do not appear to be capable of being suggested or sustained by any recognised mode of generating lines by human or finite agency, unless indeed we might suppose the lines to be material and rigid, and to be arbitrarily thrown upon the plane. The system of random lines thus laid down by Mr. Crofton has however this evident recommendation, that, as an ideal matrix of an arbitrary line on a plane of indefinite extent, it presents the most convenient basis for mathematical deductions.

* See Reprint, Vol. VII., p. 85.

2676. (Proposed by Dr. SALMON, F.R.S.)—Let the sides of the triangle of reference subtend at a point $(\alpha', \beta', \gamma')$ angles $(\theta, \theta', \theta'')$. Then

$$\sin\theta\left(\frac{\alpha}{\alpha'}\right)^{\frac{1}{2}} + \sin\theta'\left(\frac{\beta}{\beta'}\right)^{\frac{1}{2}} + \sin\theta''\left(\frac{\gamma}{\gamma'}\right)^{\frac{1}{2}} = 0$$

is a conic touching the three lines and having $(\alpha', \beta', \gamma')$ for focus. Also

$$\tan\tfrac{1}{2}\theta\left(\frac{\alpha'}{\alpha}\right) + \tan\tfrac{1}{2}\theta'\left(\frac{\beta'}{\beta}\right) + \tan\tfrac{1}{2}\theta''\left(\frac{\gamma'}{\gamma}\right) = 0$$

is a conic circumscribing the triangle and having $(\alpha', \beta', \gamma')$ for focus.

If we denote by λ, μ, ν the perpendiculars drawn from the vertices of the triangle of reference on any tangent, and ρ, ρ', ρ'' the distances of $(\alpha', \beta', \gamma')$ from the vertices; then

$$\sin\theta\left(\frac{\rho}{\lambda}\right) + \sin\theta'\left(\frac{\rho'}{\mu}\right) + \sin\theta''\left(\frac{\rho''}{\nu}\right) = 0$$

is the tangential equation of the first of these conics, and

$$\sin\tfrac{1}{2}\theta\left(\frac{\lambda}{\rho}\right) + \sin\tfrac{1}{2}\theta'\left(\frac{\mu}{\rho'}\right)^{\frac{1}{2}} + \sin\tfrac{1}{2}\theta''\left(\frac{\nu}{\rho''}\right)^{\frac{1}{2}} = 0$$

is the tangential equation of the second.

Solution by W. S. MCCAY, B.A.

If Σ be the tangential equation of a conic whose foci are $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$, then, Ω being the tangential equation of the circular points at infinity,

$$\Omega - \Sigma = (\lambda\alpha' + \mu\beta' + \nu\gamma')(\lambda\alpha'' + \mu\beta'' + \nu\gamma'')$$

the foci being intersections of common tangents.

Hence, Σ being $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2$, we have

$$\begin{array}{ll|ll} \alpha'\alpha'' = 1-A\ldots(1) & & \beta'\gamma'' + \gamma'\beta'' = 2(-F-\cos A)\ldots(4) \\ \beta'\beta'' = 1-B\ldots(2) & & \alpha'\gamma'' + \gamma'\alpha'' = 2(-G-\cos B)\ldots(5) \\ \gamma'\gamma'' = 1-C\ldots(3) & & \alpha'\beta'' + \beta'\alpha'' = 2(-H-\cos C)\ldots(6). \end{array}$$

Eliminating α'', β'', γ'' from the second three by the first three, we get three equations like

$$\beta'^2 + 2\beta'\gamma'\cos A + \gamma'^2 = B\gamma'^2 - 2F\beta'\gamma' + C\beta'^2$$

or

$$\rho^2 \sin^2 A = \beta\gamma'^2 - 2F\beta'\gamma' + C\beta'^2.$$

Now, if the triangle of reference circumscribe the conic, $A = B = C = 0$. Hence (since $a\alpha' = \rho'\rho'' \sin\theta$) we have

$$\begin{aligned} F : G : H &= \rho^2 \sin^2 A \,.\, \alpha' : \rho'^2 \sin^2 B \,.\, \beta' : \rho''^2 \sin^2 C \,.\, \gamma \\ &= \rho \sin A \sin\theta : \rho' \sin B \sin\theta' : \rho'' \sin C \sin\theta''. \end{aligned}$$

Hence

$$\frac{\rho \sin A \sin\theta}{\lambda} + \frac{\rho' \sin B \sin\theta'}{\mu} + \frac{\rho'' \sin C \sin\theta''}{\gamma} = 0,$$

or calling λ, μ, ν the perpendiculars, as Dr. Salmon does,

$$\frac{\rho \sin\theta}{\lambda} + \frac{\rho' \sin\theta'}{\mu} + \frac{\rho'' \sin\theta''}{\gamma} = 0,$$

and of course the first equation follows readily.

Again, if the triangle be inscribed in the conic,

$$\begin{array}{l|l} BC = F^2 & \rho \sin A = \gamma'\sqrt{B} + \beta'\sqrt{C} \\ AC = G^2 & \rho' \sin B = \alpha'\sqrt{C} + \gamma'\sqrt{A} \\ AB = H^2 & \rho'' \sin C = \beta'\sqrt{A} + \alpha'\sqrt{B}. \end{array}$$

Hence
$$\begin{aligned} \sqrt{A} : \sqrt{B} : \sqrt{C} &= \alpha' (-\rho \sin A\alpha' + \rho' \sin B\beta' + \rho' \sin C\gamma') \\ &: \beta' (\rho \sin A\alpha' - \rho' \sin B\beta' + \rho'' \sin C\gamma') \\ &: \gamma' (\rho \sin A\alpha' + \rho' \sin B\beta' - \rho'' \sin C\gamma'), \end{aligned}$$

therefore
$$\begin{aligned} \sqrt{A} : \sqrt{B} : \sqrt{C} &= \alpha' (-\sin\theta + \sin\theta' + \sin\theta'') \\ &: \beta' (\sin\theta - \sin\theta' + \sin\theta'') : \gamma' (\sin\theta + \sin\theta' - \sin\theta') \\ &= \alpha' \tan\tfrac{1}{2}\theta : \beta' \tan\tfrac{1}{2}\theta' : \gamma' \tan\tfrac{1}{2}\theta''; \end{aligned}$$

and the tangential equation is

$$\sqrt{(\alpha' \tan\tfrac{1}{2}\theta\lambda)} + \sqrt{(\beta \tan\tfrac{1}{2}\theta'\mu)} + \sqrt{(\gamma' \tan\tfrac{1}{2}\theta''\nu)} = 0,$$

which gives Dr. Salmon's second equation, and his fourth comes by putting λa, μb, νc for λ, μ, ν.

In the same way we find the equation, if the triangle be self-conjugate, to be

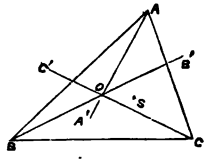
$$\tan\theta \left(\frac{\alpha}{\alpha'}\right)^2 + \tan\theta' \left(\frac{\beta}{\beta'}\right)^2 + \tan\theta'' \left(\frac{\gamma}{\gamma'}\right)^2 = 0.$$

[Dr. SALMON remarks that the theorems are obtained immediately by taking the reciprocals of the ordinary trilinear equations of a circle inscribed in a triangle and circumscribing it, using the principle that the ratio of the distances of any point from a line and from the origin of reciprocation is equal to the corresponding quantity in the reciprocal figure.]

2675. (Proposed by Professor SYLVESTER.)—If two triangles lie in perspective, prove that a cubic curve may be drawn through the six angles, such that for each triangle the tangents at its angles shall all three meet together on the curve.

Solution by F. D. THOMSON, M.A.

Let ABC, A′B′C′ be the two triangles; and let AA′, BB′, CC′ meet in O. Then we can draw a cubic through ABCA′B′C′O so as to satisfy the *two* conditions that the tangents at ABC shall meet in a point (S suppose) which shall be upon the curve.



In order to prove Professor SYLVESTER'S theorem, it will be sufficient to show that the tangents to this cubic at A′B′C′ meet in a point (S′ suppose) upon the curve.

It is convenient (as in the *Messenger of Mathematics*, Vol. III., p. 15) to call the point where the chord joining two given points on a cubic meets the

curve again, the *satellite* of these two points; or, if the two points coincide, the *satellite* of the point of contact of the tangent; *e.g.*, S is the *satellite* of A, B, or C.

Then, as in the paper referred to, since AOA′ is a straight line, the *satellites* of AOA′ are in a straight line.

Let X be the *satellite* of O, a' of A′, b' of B′, c' of C′. Then SXa' is a straight line, and similarly SXb', SXc' are straight lines; therefore $a'b'c'$ coincide in a point (S′ suppose) on the curve.

[In Professor SYLVESTER'S system of nomenclature, the Satellite is termed specifically the Connective, generically the Residual, of the two points to which it appertains. The Connective of a reiterated point is its Tangential.]

2646. (Proposed by Capt. A. R. CLARKE, R.E., F.R.S.)—A cube being cut by any plane, what is the chance that the section is a hexagon?

Solution by the PROPOSER.

Let the origin of coordinates be at the centre of the cube and the axes parallel to its sides. Imagine a sphere of radius indefinitely greater than the side of the cube, concentric with the cube. Let the positive directions of the axes of coordinates meet the sphere in X, Y, Z. A little consideration will show that we may limit the cutting planes under consideration to such as have their normals falling within the triangle XYZ. Further, if U, V, W be the middle points of ZY, XZ, YX, and the arcs XU, YV, ZW meet in O, it will be seen that we may limit the cutting planes to such as have their normals falling within any one of the six equal triangles into which XYZ is divided. We shall consider then only such cutting planes as have their normals falling within OVZ. Let e be that angle of the cube which is on the line joining the centre of the cube with O; a, b, c the angles contiguous to e, so that ae, be, ce are parallel to the positive directions of the axes; d is the fourth angle of the face eab parallel to the plane of XY. Let any cutting plane (P), whose normal meets the sphere in P (in OVZ), cut the edges ea, eb, ec in f, g, h, then $eg > ef > eh$, so that if the plane do not cut ec it does not cut ea or eb. Let $\alpha, \beta, \gamma, \delta, \epsilon$ be the distances of a, b, c, d, e from a plane parallel to the plane P, and passing through the centre of the cube. Then, if the side of the cube $= 2$, and if l, m, n be direction cosines of the normal to P, we have

$$\alpha = -l+m+n,\ \beta = l-m+n,\ \gamma = l+m-n,\ \delta = -l-m+n,\ \epsilon = l+m+n.$$

Here we see that $\delta = -\gamma$. But if we confine our attention, as we may, to such cutting planes as lie on that side of the centre of the cube which is towards the point P on the sphere, then to get an hexagonal section, γ must be positive, and δ negative. When $\delta = \gamma = 0$, $l+m = n$, and the locus of this equation is the arc of a great circle which joins UV, meeting OZ in T. In order then to have an hexagonal section, the point P must be within the triangle OTV.

Let the great circle through Z and P cut TV, OV in p and q, and let $ZP = \phi$, $VZP = \theta$, $Zp = \phi_1$, $Zq = \phi'$; then $\cot \phi_1 = \cos\theta + \sin\theta$, and $\cot \phi' = \cos\theta$; hence the ratio of the number of hexagonal sections to the

total number of sections, that is, the probability that a chance section is a hexagon, is

$$p = \frac{N}{D}, \text{ where } N = \int_0^{\frac{1}{2}\pi} \int_{\phi_1}^{\phi'} \gamma \sin\phi \, d\theta \, d\phi, \quad D = \int_0^{\frac{1}{2}\pi} \int_0^{\phi'} \epsilon \sin\phi \, d\theta \, d\phi.$$

Now we have, $l = \cos\theta \sin\phi$, $m = \sin\theta \sin\phi$, $n = \cos\phi$; therefore

$$2N = \int_0^{\frac{1}{2}\pi} \int_{\phi_1}^{\phi'} \left[(1 - \cos 2\phi)(\cos\theta + \sin\theta) - 2 \sin\phi \cos\phi \right] d\theta \, d\phi$$

$$= \int_0^{\frac{1}{2}\pi} \left[(\phi' - \phi_1)(\cos\theta + \sin\theta) - \frac{\cos\theta \sin\theta}{1 + \cos^2\theta} \right] d\theta$$

$$= \left[(\phi' - \phi_1)(-\cos\theta + \sin\theta) \right]_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} \frac{2d\theta}{1 + \cos^2\theta} + \int_0^{\frac{1}{2}\pi} \frac{3d\theta}{2 + \sin 2\theta}$$

$$= \sqrt{3} \cot^{-1}\sqrt{3} - \sqrt{2} \cot^{-1}\sqrt{2};$$

$$2D = \int_0^{\frac{1}{2}\pi} \left[(\cos\theta + \sin\theta)(\phi' - \sin\phi' \cos\phi') - \cos^2\phi' + 1 \right] d\theta$$

$$= \left[\phi'(\sin\theta - \cos\theta) \right]_0^{\frac{1}{2}\pi} = \frac{\pi}{4}.$$

Hence the chance that the section is hexagonal is

$$\frac{\sqrt{3} \cot^{-1}\sqrt{3} - \sqrt{2} \cot^{-1}\sqrt{2}}{\frac{1}{4}\pi}, \text{ which is about } 0{\cdot}04646....$$

NOTE ON THE SOLUTION OF QUESTION 2483. BY ARTEMAS MARTIN.

Mr. INGLEBY [*Reprint*, Vol. IX., p. 84] finds $65^2 = 33^2 + 56^2 = 25^2 + 60^2$, and then says :—"All other cases of the separation of 65^2 are imaginary."

I desire to add that there are two other possible cases.

Taking $p_3 = 8$, $q_3 = 1$, and $r_3 = 1$, we have $65^2 = 63^2 + 16^2$; and taking $p_4 = 2$, $q_4 = 1$, and $r_4 = 13$, we have $65^2 = 39^2 + 52^2$.

2413. (Proposed by Professor MANNHEIM.)—Three planes at right angles to each other, and intersecting in a point A, touch an ellipsoid whose centre is B. Prove that the connector AB and the normals at the three points of contact are generators of a hyperboloid.

Solution by the REV. J. WOLSTENHOLME, M.A.

Supposing the ellipsoid to rest supported by the planes: one position of equilibrium will be when its centre is vertically above A, since its centre is then

in the highest possible position of its spherical locus. The ellipsoid is then kept at rest by gravity along AB and the pressures of the planes at the points of contact along the normals. Let any straight line be drawn meeting the three normals, and take moments of the four forces about it. The moments of the three reactions are zero, because their lines of action intersect it; and the sum of the moments is zero, because there is equilibrium. Hence the moment of gravity is zero; or any straight line drawn intersecting the three normals will intersect AB. The three normals and AB must therefore be generators of the same system of a hyperboloid.

2518. (Proposed by the Rev. W. ROBERTS, M.A.)—1. Let P be a point on a given plane curve (A), O a fixed point, and Q the vertex of an equilateral hyperbola, whose centre is O, and which touches the given curve at P. Prove that the tangent at Q to the curve which is the locus of Q makes with OQ an angle equal to that made with OP by the tangent to (A) at P.

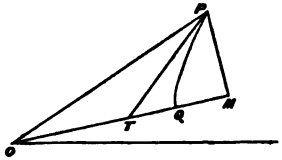
2. The curve (B), which is the inverse of the locus of Q with respect to the origin O, will be reciprocal to the given curve (A); that is to say, if (B) be regarded as the given curve, the original curve (A) will be derived from it in precisely the same manner as it has been derived from (A).

[A conic section referred to its centre, and an ellipse of Cassini, are reciprocal curves of this kind.]

Solution by A. W. PANTON, B.A.

1. Let (r, θ) be the polar coordinates of P, (ρ, ϕ) those of Q, PT the tangent at P, and PM perpendicular on OQ.

Now, by a simple property of the equilateral hyperbola, we have angle MPT = POM. Let OPT = ϵ,



then $\epsilon + 2(\theta - \phi) = 90°$,

and $\cot 2(\theta-\phi) = \tan \epsilon = r\dfrac{d\theta}{dr}$.

Again, from the polar equation of the equilateral hyperbola, $r^2 \cos 2(\theta-\phi) = \rho^2$,

therefore $r \cos 2(\theta-\phi)\, dr - \sin 2(\theta-\phi)\, r^2 (d\theta - d\phi) = \rho d\rho$.

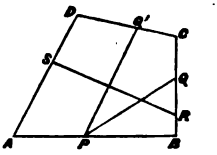
Dividing by $\sin 2(\theta-\phi)$ we have $r^2 dr \dfrac{d\theta}{d\phi} - r^2(d\theta - d\phi) = \dfrac{\rho d\rho}{\sin 2(\theta-\phi)}$,

whence $\rho^2 d\phi = \rho d\rho \cot 2(\theta-\phi)$; therefore $\rho \dfrac{d\phi}{d\rho} = \cot 2(\theta-\phi) = \tan \epsilon$.

2. Change ρ into ρ^{-1} and r into r^{-1}, then $\rho^2 \cos 2(\theta-\phi) = r^2$, which shows that the inverse of (A) is got from (B) as the inverse of (B) is from (A).

2585. (Proposed by S. WATSON.)—Four points are taken at random on the sides of a quadrilateral; what is the chance of the lines *joining* them intersecting?

Solution by the PROPOSER.



Let ABCD be the quadrilateral, and take first two of the points P, Q on contiguous sides, as AB, BC. Put $AB=a$, $BC=b$, $CD=c$, $DA=d$, $BP=x$, $BQ=y$. Then if R, S be the other points, in order that PQ and RS may intersect, when R lies upon BQ, S must lie on $AP+AD+DC = c+d+a-x$, and when R lies on BP, S must lie on $AD+DC+CQ = c+d+b-y$; hence when P and Q take every position on AB, BC, the number of favourable cases is expressed by

$$\int_0^b \int_0^a \left\{(c+d+a-x)y+(c+d+b-y)x\right\} dy\,dx$$

$$= \tfrac{1}{2}ab\left\{ab+(a+b)(c+d)\right\} \ldots\ldots\ldots\ldots (1).$$

By symmetry, when P and Q lie on BC and CD, on CD and DA, and on DA and AB, the number of favourable positions is respectively

$$\tfrac{1}{2}bc\left\{bc+(b+c)(d+a)\right\}, \qquad \tfrac{1}{2}cd\left\{cd+(c+d)(a+b)\right\},$$

$$\tfrac{1}{2}da\left\{da+(d+a)(b+c)\right\} \ldots\ldots\ldots\ldots (2, 3, 4).$$

Let now Q be on the side opposite to P, as at Q', and put $CQ'=z$; then according as R lies on BP, BC, CQ', the positions of S must be $c+d-z$, $a+c+d-x-z$, $d+a-x$; hence the favourable cases, when P and Q' take every position on AB, CD, are

$$\int_0^c \int_0^a \left\{(c+d-z)x+b(a+c+d-x-z)+(d+a-x)z\right\} dz\,dx$$

$$= \tfrac{1}{2}ac\left\{ac+2bd+(a+c)(b+d)\right\} \ldots\ldots\ldots\ldots (5),$$

and similarly, when P and Q' lie on BC, DA, we have

$$\tfrac{1}{2}bd\left\{bd+2ac+(a+c)(b+d)\right\} \ldots\ldots\ldots\ldots (6).$$

Now (1).... (6) must be all multiplied by 4, because P and Q, as also R and S, may be interchanged; hence the required chance is

$$p = \frac{4\left\{(1)+(2)\ldots+(6)\right\}}{(a+b+c+d)^4} = \frac{2(ab+ac+ad+bc+bd+cd)^2-8abcd}{(a+b+c+d)^4}.$$

In the case of a parallelogram, we have $c=a$, $d=b$, and then

$$p = \frac{(a^2+4ab+b^2)^2-4a^2b^2}{8(a+b)^4}.$$

In the case of a rhombus, we have $p = \frac{1}{4}$.

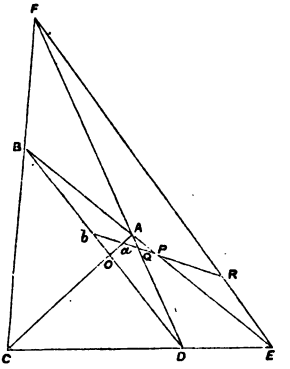
2669. (Proposed by J. J. WALKER, M.A.)—If the diagonals d, d' of any plane quadrilateral be divided at their point of intersection in the ratios $p : p'$ and $q : q'$ respectively, and θ be the angle between them, then (1) the square of the third diagonal will be equal to

$$\frac{p^2p'^2(q^2-q'^2)^2\,d^2+q^2q'^2(p^2-p'^2)^2\,d'^2+2pp'\,qq'\,(p^2-p'^2)(q^2-q'^2)\,dd'\cos\theta}{(pq-p'q')^2(pq'-p'q)^2};$$

and (2) the line which divides these diagonals in the ratios $m : m'$ and $n : n'$ respectively will divide the third diagonal in the ratio

$$\frac{mnp'q' - m'n'pq}{p'q' - pq} : \frac{m'npq' - mn'p'q}{pq' - p'q}.$$

Solution by S. BILLS; *the* PROPOSER; *and others.*



1. In the accompanying diagram let ABCDEF be a complete quadrilateral; AC, BD, EF being the three diagonals, the first two intersecting in O.

Let $AC = d$ and $BD = d'$; then, by the question, $AO = \frac{dp}{p+p'}$, $CO = \frac{dp'}{p+p'}$, $BO = \frac{d'a}{q+q'}$, $DO = \frac{d'q'}{q+q'}$; also, taking a point a in AC so that $Aa : Ca = m : m'$, and a point b in BD so that $Bb : Db = n : n'$, we shall have

$$Aa = \frac{dm}{m+m'}, \quad Ca = \frac{dm'}{m+m'};$$

$$Bb = \frac{d'n}{n+n'}, \quad Db = \frac{d'n'}{n+n'}.$$

For shortness, put $\frac{p}{p+p'} = u$, $\frac{q}{q+q'} = v$, $\frac{m}{m+m'} = r$, $\frac{n}{n+n'} = s$;

then, taking O for origin, and OA, OB for axes of x and y, the equations of the several lines will be as follows,—

$$AB \ldots\ldots\ldots\ldots\ldots \frac{x}{du} + \frac{y}{d'v} = 1; \quad BC \ldots\ldots - \frac{x}{d(1-v)} + \frac{y}{d'v} = 1;$$

$$CD \ldots\ldots \frac{x}{d(1-u)} + \frac{y}{d'(1-v)} = -1; \quad DA \ldots\ldots \frac{x}{du} - \frac{y}{d'(1-v)} = 1;$$

$$ab \ldots\ldots \frac{x}{d(u-r)} + \frac{y}{d'(v-s)} = 1.$$

Let ba produced meet BA in P, AD in Q, and EF in R. Let (x_E, y_E) be the coordinates of the point E; then we readily find

$$x_E = \frac{du(1-u)}{v-u}, \quad y_E = \frac{d'v(1-v)}{u-v}.$$

Using a similar notation for the other points, we have

$$x_F = \frac{du(1-u)}{1-u-v}, \quad y_F = \frac{d'v(1-v)}{1-u-v}; \quad x_P = \frac{dus(u-r)}{us-vr}, \quad y_P = \frac{d'vr(v-s)}{vr-us};$$

$$x_Q = \frac{du(u-r)(1-s)}{(1-s)u-(1-v)r}, \quad y_Q = \frac{d'(1-v)(v-s)r}{(1-v)r-(1-s)u}.$$

Now $EF^2 = (x_E - x_F)^2 + (y_E - y_F)^2 - 2(x_E - x_F)(y_E - y_F)\cos\theta$ ..... (α).

But $x_E - x_F = du(1-u)\left(\frac{1}{v-u} - \frac{1}{1-u-v}\right) = \frac{dpp'(q'^2-q^2)}{(p'q-pq')(p'q'-pq)}$,

and $y_E - y_F = dv(1-v)\left(\frac{1}{u-v} - \frac{1}{1-u-v}\right) = \frac{d'qq'(p^2-p'^2)}{(p'q-pq')(p'q'-pq)}$.

Substituting the above results in (α), we have

$$EF^2 = \frac{p^2p'^2(q^2-q'^2)^2d^2 + q^2q'^2(p^2-p'^2)^2d'^2 - 2pp'qq'(p^2-p'^2)(q^2-q'^2)\cos\theta}{(pq-p'q')^2(pq'-p'q)^2}$$

2. Again, $AP : EP = x_P - x_A : x_E - x_P$,

and $AQ : FQ = x_Q - x_A : x_F - x_Q$.

But, by a well-known theorem, we have

$ER : FR = EP.AQ : FQ.AP$

$= (x_E - x_P)(x_Q - x_A) : (x_P - x_A)(x_F - x_Q)$ ..... (β).

Substituting (in β) for x_E, x_F, x_P, &c., their values, as given above, we find

$$ER : FR = \frac{m'npq' - mn'p'q}{pq' - p'q} : \frac{mnp'q' - m'n'pq}{p'q' - pq}.$$

If $m = m'$ and $n = n'$, it is evident from the above that $ER = FR$; which shows that the middle points of the three diagonals of a complete quadrilateral lie in a straight line, as is well known from other considerations.

2662. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Prove that

$$\frac{\sin n\alpha \sin(\beta-\gamma) + \sin n\beta \sin(\gamma-\alpha) + \sin n\gamma \sin(\alpha-\beta)}{\sin(\gamma-\beta) + \sin(\beta-\alpha) + \sin(\alpha-\gamma)} = 2\Sigma(p\alpha + q\beta + r\gamma);$$

p, q, r being positive integers whose sum is n, no two being zero, and the coefficient when one is zero being 1 instead of 2.

Solution by MORGAN JENKINS, M.A.

By the ordinary formulæ we have

$$2\sum_{q=1}^{q=n-p-1} \sin\left\{p\alpha + q\beta + (n-p-q)\gamma\right\}$$

$$= \frac{\cos\left\{p\alpha + (n-p)\gamma + \frac{1}{2}(\beta-\gamma)\right\} - \cos\left\{p\alpha + (n-p)\beta - \frac{1}{2}(\beta-\gamma)\right\}}{\sin\frac{1}{2}(\beta-\gamma)}$$

$$= \frac{\cos\left\{p\alpha + (n-p)\gamma + \frac{1}{2}(\beta-\gamma)\right\}}{\sin\frac{1}{2}(\beta-\gamma)} + \text{similar terms formed from transposition of } \beta \text{ and } \gamma.$$

Again, summing for the variation of p between 1 and $n-2$, we shall obtain for the sum of the series of terms which have the coefficient 2,

$$\frac{\cos(n\alpha-\alpha+\gamma)+\text{sym. terms}-\cos(n\alpha-\alpha+\beta)-\text{sym. terms}}{4\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)} \quad \ldots\ldots (1).$$

Also, summing $\sum_{p=1}^{p=n-1}\sin\{p\alpha+(n-p)\beta\}$ and the symmetrical expressions, we shall obtain

$$\frac{-\cos(n\alpha-\alpha+\gamma)-\text{sym. terms}+\cos(n\alpha-\alpha+\beta)+\text{sym terms}}{4\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)}$$

$$+\frac{\frac{1}{2}\{\cos(n\alpha-\beta+\gamma)-\cos(n\alpha+\beta-\gamma)\}+\text{sym. terms}}{4\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)} \quad \ldots\ldots\ldots (2).$$

Adding (1) to (2), the numerator is equal to the numerator of the given left-hand expression, and the denominator of (1) and (2) to the denominator of the same given expression.

2587. (Proposed by S. ROBERTS, M.A.) — Writing $F(z)$ for $a_0+a_1z+a_1\frac{z^2}{1.2}+\frac{a_2.z^3}{1.2.3}+$ &c., show that the coefficient of $x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}$ in $F(x_1-x_2).F(x_2-x_3)\ldots.F(x_m-x_1)$ is an invariant, or the leading term of a covariant relative to the function formed by taking the terms of $F(z)$ involved in the coefficient and substituting binomial for exponential coefficients.

Solution by the PROPOSER.

The coefficient of $x^{k_1}\ldots x^{k_m}$ must be formed from the terms of

$$\pm\Sigma\frac{a_{a_1}.a_{a_2}\ldots a_{a_m}}{[a_1][a_2]\ldots[a_m]}(x_1-x_2)^{a_1}(x_2-x_3)^{a_2}\ldots(x_m-x_1)^{a_m},$$

where $a_1+a_2+\ldots+a_m=k_1+k_2+\ldots+k_m$. Suppose, then, we take the product $\pm\frac{a_{a_1}.a_{a_2}\ldots a_{a_m}}{[p_1][a_1-p_1]\ldots[p_m][a_m-p_m]}x_1^{p_1}.x_2^{a_1-p_1}\ldots x_m^{p_m}.x_1^{a_m-p_m}$, we must have

$$p_1+a_m-p_m=k_1,\quad p_2+a_1-p_1=k_2,\quad \ldots\quad p_m+a_{m-1}-p_{m-1}=k_m;$$

and the coefficient in question is

$$\pm\Sigma\frac{a_{k_1+p_m-p_1}.a_{k_2+p_1-p_2}\ldots a_{k_m+p_{m-1}-p_m}}{[p_1][k_2-p_1][p_2][k_3-p_2]\ldots[p_m][k_1-p_1]} \quad \ldots\ldots\ldots (a).$$

But by Professor CAYLEY'S method we may find a covariant of a function with binomial coefficients by operating with

$$(D_{x_1} D_{y_2} - D_{x_2} D_{y_1})^{k_1} (D_{x_2} D_{y_3} - D_{x_3} D_{y_2})^{k_2} \dots (D_{x_m} D_{y_1} - D_{x_1} D_{y_m})^{k_m}.$$

If we divide out the factor $[k_1]\,[k_2] \dots [k_m]$ which appears in the development, the coefficient of

$$\pm (D_{x_1} D_{y_2})^{k_1 - p_1} (D_{x_2} D_{y_1})^{p_1} \dots\dots (D_{x_m} D_{y_1})^{k_m - p_m} (D_{x_1} D_{y_m})^{p_m}$$

is $$\pm \frac{1}{[p_1]\,[k_1 - p_1]\,[p_2]\,[k_2 - p_2] \dots [p_m]\,[k_m - p_m]}.$$

Also the corresponding argument in the leading term is

$$a_{k_1 - p_1 + p_m} \cdot a_{k_2 - p_2 + p_1} \dots\dots a_{k_m - p_m + p_{m-1}}.$$

This result agrees with (α). Hence the coefficient named in the question is the leading term of the covariant $(1.2)^{k_1} (2.3)^{k_2} \dots (m.1)^{k_m}$, according to the Hyperdeterminant notation.

2647. (From WOLSTENHOLME'S *Book of Mathematical Problems.*)—In the equation $x^3 - qx + r = 0$ it is known that q and r both lie between $+1$ and -1; assuming all values between these limits to be equally probable, prove that the chance of all the roots of the equation being real is $\frac{2}{45}\sqrt{3}$.

Solution by J. J. WALKER, M.A.

The chance that q should have any one particular value between the assigned limits is $\frac{1}{2}dq$; that $(\frac{1}{2}r)^2$ should not be greater than $(\frac{1}{3}q)^3$, or r lie between $\pm 2\,(\frac{1}{3}q)^{\frac{3}{2}}$ is $4(\frac{1}{3}q)^{\frac{3}{2}} \div 2 = 2\,(\frac{1}{3}q)^{\frac{3}{2}}$. The chance then that q should be positive and $(\frac{1}{2}r)^2$ not greater than $(\frac{1}{3}q)^{\frac{3}{2}}$, the conditions necessary that the roots of $x^3 - qx + r = 0$ should be all real, is

$$\frac{1}{\sqrt{27}} \int_0^1 q^{\frac{3}{2}}\, dq = \frac{2}{5\sqrt{27}} = \frac{2\sqrt{3}}{45}.$$

2428. (Proposed by the Rev. J. BLISSARD.)—Prove that

(1) $$\dots\dots \frac{\Gamma(mx)}{\Gamma(nx)}\,(x=0) = \frac{n}{m};$$

(2) $$\dots\dots \frac{\Gamma(n)}{\Gamma(2n)} - \frac{\Gamma(n+2)}{\Gamma(2n+2)} \cdot \frac{\pi^2}{1.2} + \frac{\Gamma(n+4)}{\Gamma(2n+4)} \cdot \frac{\pi^4}{1.2.3.4} - \&c. = 0.$$

Solution by the PROPOSER.

1. Since $\log \Gamma(1+x) = -\gamma x + \frac{1}{2}S_2 x^2 - \&c.$, [See DE MORGAN'S *Differential and Integral Calculus*, p. 585,] it is evident that

$$\Gamma(1+x) = 1 + \text{terms in ascending powers of } x. \quad \text{Hence}$$

$$\frac{\Gamma(1+mx)}{\Gamma(1+nx)}\left(= \frac{mx\,\Gamma(mx)}{nx\,\Gamma(nx)} = \frac{m}{n}\cdot\frac{\Gamma(mx)}{\Gamma(nx)}\right) = 1 + \text{terms in ascending powers of } x,$$

therefore $\quad \frac{m}{n}\frac{\Gamma(mx)}{\Gamma(nx)}(x=0) = 1$ and $\frac{\Gamma(mx)}{\Gamma(nx)}(x=0) = \frac{n}{m}$.

2. In *Reprint*, Vol. III., p. 87, the following theorem is proved:—

$$fx - \frac{n}{m+n} f_1 x \cdot \frac{x}{1} + \frac{n(n+1)}{(m+n)(m+n+1)} f_2 x \cdot \frac{x^2}{1.2} - \&c.$$

$$= f0 + \frac{m}{m+n} f_1 0 \cdot \frac{x}{1} + \frac{m(m+1)}{(m+n)(m+n+1)} f_2 0 \cdot \frac{x^2}{1.2} + \&c. \ldots\ldots \text{(I.)}$$

Let $m = n$, and multiply both sides by $\frac{1.2\ldots(n-1)}{1.2\ldots(2n-1)}$, then

$$\frac{\Gamma(n)}{\Gamma(2n)} fx - \frac{\Gamma(n+1)}{\Gamma(2n+1)} \cdot f_1 x \cdot \frac{x}{1} + \frac{\Gamma(n+2)}{\Gamma(2n+2)} \cdot f_2 x \cdot \frac{x^2}{1.2} - \&c.$$

$$= \frac{\Gamma(n)}{\Gamma(2n)} f0 + \frac{\Gamma(n+1)}{\Gamma(2n+1)} \cdot f_1 0 \cdot \frac{x}{1} + \frac{\Gamma(n+2)}{\Gamma(2n+2)} \cdot f_2 0 \cdot \frac{x^2}{1.2} + \&c.$$

Now let $fx = \cos x$, then after slight reduction, we have

$$(1-\cos x)\left(\frac{\Gamma(n)}{\Gamma(2n)} - \frac{\Gamma(n+2)}{\Gamma(2n+2)} \cdot \frac{x^2}{1.2} + \frac{\Gamma(n+4)}{\Gamma(2n+4)} \cdot \frac{x^4}{1.2.3.4} - \&c.\right)$$

$$= \sin x \left(\frac{\Gamma(n+1)}{\Gamma(2n+1)} \cdot \frac{x}{1} - \frac{\Gamma(n+3)}{\Gamma(2n+3)} \cdot \frac{x^3}{1.2.3} + \&c.\right)$$

Let $x = \pi$, then $\frac{\Gamma(n)}{\Gamma(2n)} - \frac{\Gamma(n+2)}{\Gamma(2n+2)} \cdot \frac{\pi^2}{1.2} + \frac{\Gamma(n+4)}{\Gamma(2n+4)} \cdot \frac{\pi^4}{1.2.3.4} - \&c. = 0.$

Let $x = \frac{\pi}{2}$, then $\frac{\Gamma(n)}{\Gamma(2n)} - \frac{\Gamma(n+1)}{\Gamma(2n+1)} \cdot \frac{\pi}{2} - \frac{\Gamma(n+2)}{\Gamma(2n+2)} \cdot \frac{1}{1.2}\left(\frac{\pi}{2}\right)^2 + \&c. = 0.$

In (I) put $-n$ for n, and then put $m = n+1$, then we have

$$fx + \frac{n}{1} \cdot f_1 x \cdot \frac{x}{1} + \frac{n(n-1)}{1.2} f_2 x \cdot \frac{x^2}{1.2} + \&c.$$

$$= f0 + \frac{n+1}{1} f_1 0 \cdot \frac{x}{1} + \frac{(n+1)(n+2)}{1.2} \cdot f_2 0 \cdot \frac{x^2}{1.2} + \&c. \ldots\ldots \text{(II.)}$$

From this general formula, a great variety of novel and interesting results may (especially by the use of the Representative Notation) be readily obtained.

2568. (Proposed by Professor BALL.)—Express the roots of the equation $a^6x^6 + 18a^4(bd-c^2)x^4 + 81a^2(bd-c^2)^2x^2 + 27d^2(a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd) = 0$, in terms of α, β, γ the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$.

Solution by J. J. WALKER, M.A.

The roots of the proposed equation are at once seen to be the differences of the reciprocals of those of the cubic.

2571. (Proposed by Professor BALL.)—The equation whose roots are the ratios of the roots of $ax^3+3bx^2+3cx+d=0$, is

$$a^2d^2x^6+(3a^2d^2-9abcd)\,x^5+(6a^2d^2-45abcd+27ac^3+27b^3d)\,x^4$$
$$+(7a^2d^2-54abcd+54ac^3-81b^2c^2+54b^3d)\,x^3$$
$$+(6a^2d^2-45abcd+27ac^3+27b^3d)\,x^2+(3a^2d^2-9abcd)\,x+a^2d^2=0.$$

Prove this, and show that its discriminant is

$$a^4d^4\,(ad-9bc)^2\,(b^3d-ac^3)^4\,(a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd)^3;$$

and account *à priori* for the circumstance that, while in the equation b and c rise to the third power, a and d only reach the second.

Solution by J. J. WALKER, M.A.

The sum of the six ratios $\frac{\alpha}{\beta}, \frac{\alpha}{\gamma} \ldots \frac{\gamma}{\beta}$ is $-\frac{a}{d}\Sigma a^2\beta$ or $3\,\frac{-ad+3bc}{ad}$.

The sum of the products taken two and two is

$$3+\Sigma\frac{\alpha}{\beta}+\frac{a}{d}\Sigma a^3+\frac{a^2}{d^2}\Sigma a^3\beta^3,$$

which becomes, after substituting for the symmetric functions their values in terms of the coefficients, and reducing,

$$\frac{3}{a^2d^2}\,(2a^2d^2-15abcd+9ac^3+9b^3d).$$

The sum of the products of the six ratios taken three and three is

$$2+2\Sigma\frac{\alpha}{\beta}+\Sigma\frac{\alpha^2}{\beta^2}=\frac{1}{a^2d^2}\,(-7a^2d^2+54abcd-54ac^3+81b^2c^2-54b^3d).$$

The transformed equation is therefore

$$a^2d^2(x^6+1)+3ad\,(ad-3bc)(x^5+x)+3\,(2a^2d^2-15abcd+9ac^2+9b^3d)(x^4+x^2)$$
$$+(7a^2d^2-54abcd+54ac^3-81b^2c^2+54b^3d)\,x^3=0.$$

The discriminant of the given equation is most readily calculated from the product of the squares of the differences of the roots. This is easily seen to be equal to

$$-\frac{1}{a^{16}\beta^{16}\gamma^{16}}\,(\alpha^2-\beta^2)^2\,(\gamma^2-\alpha^2)^2\,(\beta^2-\gamma^2)^2\,(\alpha^2-\beta\gamma)^4\,(\beta^2-\alpha\gamma)^4\,(\gamma^2-\alpha\beta)^4\,(\alpha-\beta)^4$$
$$(\gamma-\alpha)^4\,(\beta-\gamma)^4,$$

or

$$\frac{1}{a^{16}\beta^{16}\gamma^{16}}\,(\alpha+\beta)^2\,(\gamma+\alpha)^2\,(\beta+\gamma)^2\,(\alpha^2-\beta\gamma)^4\,(\beta^2-\alpha\gamma)^4\,(\gamma^2-\alpha\beta)^4\,(\alpha-\beta)^6$$
$$(\gamma-\alpha)^6\,(\beta-\gamma)^6.$$

Now $(\alpha+\beta)(\gamma+\alpha)(\beta+\gamma) = \Sigma a.\Sigma a\beta - a\beta\gamma = \frac{9bc-ad}{a^2}$,

and $(\alpha^2-\beta\gamma)(\beta^2-\alpha\gamma)(\gamma^2-\alpha\beta) = a\beta\gamma.\Sigma\alpha^3 - \Sigma\alpha^2\beta^2 = \frac{27(b^3d-ac^3)}{a^4}$;

also, $(\alpha-\beta)^2(\gamma-\alpha)^2(\beta-\gamma)^2 = -\frac{27}{a^4}\left\{(ad-bc)^2-4(b^2-ac)(c^2-bd)\right\}$.

Hence the required product of squares of differences is

$$-\frac{(27)^7}{a^{16}d^{16}}(ad-9bc)^2(ac^3-b^3d)^4\left\{(ad-bc)^2-4(b^2-ac)(c^2-bd)\right\}^3.$$

The circumstance of b and c rising to an index higher by 1 than the highest of a and d is common to all expressions which are the values of symmetric functions of the roots in terms of the coefficients, and into which those holding the same places from the two ends must enter similarly; and the reason of this is that each successive coefficient, divided by that of the highest power, is one degree higher relatively to the roots than its predecessor.

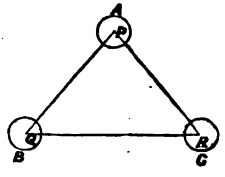
2523. (Proposed by T. SAVAGE, M.A.)—If A, B, C be three closed areas situated in such a manner that no straight line can be drawn meeting more than two of them, the average area of all triangles formed by joining three points taken at random in A, B, C respectively is equal to the area of the triangle formed by joining the centres of gravity of A, B, C.

2. If A, B, C, D be four closed volumes such that no plane can meet more than three of them, the average volume of the tetrahedron formed by joining four points taken at random in A, B, C, D respectively is equal to the volume of the tetrahedron formed by joining the centres of gravity of A, B, C, D respectively.

Solution by the REV. JOSEPH WOLSTENHOLME, M.A.

1. Take P, Q, R, three points in the areas A, B, C, and let du, dv, dw be elementary areas about P, Q, R. Then (Δ being the area of the triangle PQR) the average value of the triangle PQR

$$= \frac{1}{ABC}\iiint \Delta\, du\, dv\, dw;$$



but if z be the distance of R from PQ,

$$\int \Delta\, dw = \tfrac{1}{2}\,PQ\int z\, dw = \tfrac{1}{2}\,PQ.\bar{z}.C = C.\Delta\, PQG_3,$$

G_1, G_2, G_3 being the centres of gravity of the three areas.

So $\int \Delta\, PQG_3\, dv = \tfrac{1}{2}\,PG_3\int y\, dv$, y being the distance of Q from PG_3,

$$= \tfrac{1}{2}\,PG_3.\bar{y}.B = \Delta\, PG_2G_3.B,$$

and $\int \Delta\, PG_2G_3\, du = \tfrac{1}{2}\,G_2G_3\int x\, du = \tfrac{1}{2}\,G_2G_3.\bar{x}.A = \Delta\, G_1G_2G_3.A$,

therefore the average value of Δ is the area of the triangle $G_1G_2G_3$.

It is necessary that all the area C shall be on the same side of PQ, otherwise the distance of G_3 from PQ, though the algebraical mean of the distances of points in C from that line, is not the average here desired. The areas must therefore be so placed that no straight line can be drawn intersecting all three.

2. Exactly the same method proves the corresponding property for four volumes, under like conditions.

2497. (Proposed by S. ROBERTS, M.A.)—Show that, if $a, b, c \ldots h$ denote $\frac{1}{1.2} \cdot \frac{d^2y}{dx^2}, \frac{1}{1.2.3} \cdot \frac{d^3y}{dx^3}, \frac{1}{1.2.3.4} \cdot \frac{d^4y}{dx^4}, \ldots \frac{1}{1.2 \ldots 9} \cdot \frac{d^9y}{dx^9}$, the general differential equation of

$$a_1x^3 + a_2x^2y + b_1xy^2 + b_2y^3 + a_3x^2 + dxy + b_3y^2 + c_1x + c_2y + c_3 = 0$$

is

$$\begin{vmatrix} a, & b, & c, & a^2 \\ b, & c, & d, & 2ab \\ c, & d, & e, & 2ac + b^2 \\ d, & e, & f, & 2ad + 2bc \\ e, & f, & g, & 2ae + 2bd + c^2 \\ f, & g, & h, & 2af + 2be + 2cd, \end{vmatrix} \quad \begin{matrix} , & 0 & , & 0 \\ , & a^2 & , & 0 \\ , & 2ab & , & a^3 \\ , & 2ac + b^2 & , & 3a^2b \\ , & 2ad + 2bc & , & 3a^2c + 3ab^2 \\ , & 2ae + 2bd + c^2, & & 3a^2d + 6abc + b^3 \end{matrix} = 0$$

Solution by the PROPOSER.

This is an exercise in determinants. We have to reduce

$$\begin{vmatrix} D^4y^3, & D^4xy^2, & D^4x^2y, & D^4y^2, & D^4xy, & D^4y \\ D^5y^3, & D^5xy^2, & D^5x^2y, & D^5y^2, & D^5xy, & D^5y \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ D^9y^3, & D^9xy^2, & D^9x^2y, & D^9y^2, & D^9xy, & D^9y \end{vmatrix} = 0,$$

where D means $\frac{d}{dx}$. Making use of $D^4xy^2 = xD^4y^2 + 4D^3y^2$, &c., we see that x does not appear explicitly in the result, and the determinant may be written

$$\begin{vmatrix} D^4y^3, & D^4y^2, & 4D^3y^2, & D^4y, & 4D^3y, & 6D^2y \\ D^5y^3, & D^5y^2, & 5D^4y^2, & D^5y, & 5D^4y, & 10D^3y \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ D^9y^3, & D^9y^2, & 9D^8y^2, & D^9y, & 9D^8y, & 86D^7y \end{vmatrix} = 0 \ldots\ldots\ldots (a).$$

To reduce this further, we write the first column in the form

$$y^2D^4y + 4Dy^2\ D^3y + \ldots\ldots + D^4y^2,$$
$$y^2D^5y + 5Dy^2\ D^4y + \ldots\ldots + D^5y^2,$$
$$\text{&c.} \qquad\qquad \text{&c.}$$

Now, observing the forms of the remaining columns, we may write for the above

$$\left.\begin{array}{l} 0 \\ 10D^3y^2D^2y \\ 20D^3y^2D^3y + 15D^4y^2D^2y \\ 35D^3y^2D^4y + 35D^4y^2D^3y + 21D^5y^2D^2y \\ 56D^3y^2D^5y + 70D^4y^2D^4y + 56D^5y^2D^3y + 28D^6y^2D^2y \\ 84D^3y^2D^6y + 126D^4y^2D^5y + 126D^5y^2D^4y + 84D^6y^2D^3y + 36D^7y^2D^2y \end{array}\right\} \ldots(1).$$

Again, we may substitute $2yB + 6y'A$ for D^3y^2, $2yC + 8y'B + 6A^2$ for D^4y^2, $2yD + 10y'E + 20ab$ for $5Dy^2$, &c., where y', A, B, C, D, &c., are written for Dy, D^2y, D^3y, D^4y, D^5y, &c. These substitutions being made in the second and third columns, it is easily seen that for those columns we may write

$$\left.\begin{array}{l} 6A^2 \\ 20AB \\ 30AC + 20B^2 \\ 42AD + 70BC \\ 56AE + 112BD + 70C^2 \\ 72AF + 168BE + 252CD \end{array}\right\} (2), \qquad \left.\begin{array}{l} 0 \\ 5.\ 6A^2 \\ 6.20AB \\ 7.30AC + 20B^2 \\ 8.42AD + 70BC \\ 9.56AE + 112BD + 70C^2 \end{array}\right\} (3).$$

Making the same substitutions in (1), and comparing the result with (2) and (3), we find that for (1) may be written

$$\left.\begin{array}{l} 0 \\ 0 \\ 15A^3 \\ 105A^2B \\ 210A^2C + 28AB^2 \\ 378A^2D + 1260ABC + 280B^3 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1').$$

We have only to replace the three first columns of (a) by (1'), (2), (3), and then put $1.2a$ for A, $1.2.3b$ for B, &c. The irrelevant numerical factors being divided out, the determinant takes the form given in the question.

It will be observed that the terms of the determinant are derivable from one another in a symmetrical manner; and the result, together with the corresponding differential equations for functions of the first and second degrees (Boole, quoting Monge, *Differential Equations*, p. 20), indicates a general form by analogy.

2461. (Proposed by S. ROBERTS, M.A.)—If a curve of the nth degree be translated in one direction, the centre of mean distances of the points common to any two positions of the curve lies always on a straight line.

Solution by F. D. THOMSON, M.A.

Take the axis of x in the direction of translation, and let the equation to the curve in its *original* position be $F(x, y) = 0$ ..... (1);

and therefore in the *final* position the equation will be of the form

$$F(x+h, y) = 0, \text{ or } F(x,y) + h\frac{dF}{dx} + \frac{h^2}{\lfloor 2}\frac{d^2F}{dx^2} + \ldots + \frac{h^n}{\lfloor n}\frac{d^nF}{dx^n} F = 0 \ldots (2).$$

Hence, combining (1) and (2), we have for the points of intersection of the two curves

$$F(x,y) = 0, \text{ and } \frac{dF}{dx} + \frac{h}{\lfloor 2}\frac{d^2F}{dx^2} + \frac{h^2}{\lfloor 3}\frac{d^3F}{dx^3} + \ldots + \frac{h^{n-1}}{\lfloor n}\frac{d^nF}{dx^n} = 0 \ldots . (1, 3.)$$

We shall have to eliminate x between (1) and (3).

Now let $\frac{x}{y} = u$, and substitute for x in (1) and (3).

Then $(1) \equiv y^n f_n(u) + y^{n-1} f_{n-1}(u) + \ldots + y f_1(u) + C = 0$

(the *suffixes* serving to denote the highest power of u in the coefficients of the powers of y),

and $$(2) \equiv \frac{1}{y}\frac{dF}{du} + \frac{h}{\lfloor 2}\frac{1}{y^2}\frac{d^2F}{du^2} + \ldots + \frac{h^{n-1}}{\lfloor n}\frac{1}{y^n}\frac{d^nF}{du^n} = 0.$$

Now
$$\frac{dF}{du} = y^n f'_n(u) + y^{n-1} f'_{n-1}(u) + \ldots + y f_1'(u),$$
$$\frac{d^2F}{du^2} = y^n f''_n(u) + y^{n-1} f''_{n-1}(u) + \ldots + y^2 f_2''(u),$$
$$\&c. = \&c.,$$
$$\frac{d^nF}{du^n} = y^n f^n_n(u);$$

therefore
$$\frac{1}{y}\frac{dF}{du} = y^{n-1} f'_n(u) + y^{n-2} f'_{n-1}(u) + \ldots + f_1'(u),$$
$$\frac{h}{\lfloor 2}\cdot\frac{1}{y^2}\frac{d^2F}{du^2} = \frac{h}{2} y^{n-2} f''_n(u) + \ldots + \frac{h}{2} f_2''(u),$$
$$\&c. = \&c.,$$
$$\frac{h^{n-1}}{\lfloor n}\cdot\frac{1}{y^n}\frac{d^nF}{du^n} = \frac{h^{n-1}}{\lfloor n} f^n_n(u);$$

therefore
$$(2) \equiv y^{n-1} f'_n(u) + y^{n-2}\left\{f'_{n-1}(u) + \frac{h}{2} f''_n(u)\right\} + \&c.$$
$$+ \left\{f_1'(u) + \frac{h}{2} f_2''(u)\right\} + \&c. = 0.$$

Now all the n values of u from (1) may be shown to be given in terms of y by the expansion $u = \alpha + \frac{\beta}{y} + \frac{\gamma}{y^2} + \&c.$

Where α is one of the roots of $f_n(\alpha) = 0$,

$$\beta = -\frac{f_{n-1}(\alpha)}{f'_n(\alpha)}, \ \&c.,$$

provided that the equation $f_n(\alpha) = 0$ has no *equal* roots.

Hence, substituting for u in (2), we get

$$y^{n-1}\left\{f'_n(\alpha)+\frac{\beta}{y}f''_n(\alpha)+\ldots\right\}$$

$$+y^{n-2}\left\{f'_{n-1}(\alpha)+\frac{\beta}{y}f''_{n-1}(\alpha)+\frac{h}{2}f''_n(\alpha)+\ldots\right\}+\&c.=0,$$

or $$y^{n-1}f'_n(\alpha)+y^{n-2}\left\{\beta f''_n(\alpha)+f'_{n-1}(\alpha)+\frac{h}{2}f''_n(\alpha)\right\}+\&c.=0;$$

therefore $$\Sigma y=-\Sigma\frac{\beta f''_n(\alpha)+f'_{n-1}(\alpha)+\frac{h}{2}f''_n(\alpha)}{f'_n(\alpha)}$$

where the Σ on the right-hand side denotes that all the roots of the equation $f_n(\alpha)=0$ are to be substituted successively for α, and the results summed.

Hence $$\Sigma y=\text{constant}-\frac{h}{2}\Sigma\frac{f''_n(\alpha)}{f'_n(\alpha)}.$$

Now if $$f_n(x)=(x-\alpha)(x-\beta)(x-\gamma)\ldots.(x-\lambda),$$

$$\frac{f''_n(\alpha)}{(n-2)f'_n(\alpha)}=\frac{1}{\alpha-\beta}+\frac{1}{\alpha-\gamma}+\frac{1}{\alpha-\delta}+\ldots.+\frac{1}{\alpha-\lambda},$$

therefore $$\Sigma\frac{f''_n(\alpha)}{f'_n(\alpha)}=0,\qquad\text{therefore}\qquad\Sigma y=\text{constant},$$

therefore the centre of mean position of the points of intersection moves along a straight line parallel to the motion of translation.

Note.—The above investigation does not hold good if the terms of highest dimensions form a perfect square.

2414. (Proposed by Professor WHITWORTH.)—1. The number of ways in which n things all different can be allotted to r different persons, blank lots being admissible, is r^n.

2. If blank lots be *not* admissible, the number of ways is the coefficient of x^n in the expansion of $(e^x-1)^r\,\underline{|n}$.

3. The number of ways in which n things all alike can be allotted to r different persons, blank lots being admissible, is the number of combinations of $n+r-1$ things taken $r-1$ at a time.

4. If blank lots be *not* admissible, the number of ways is the number of combinations of $n-1$ things $r-1$ at a time.

Solution by the REV. J. WOLSTENHOLME, M.A.

1. Any one of the n things may be given to any one of the r persons, A, B, C, D.....and the number of ways is therefore r^n.

2. From the last number must be subtracted the number of ways in which A, or B, or C, ... may be void, that is $r(r-1)^n$. In performing this subtraction we have subtracted twice all the ways in which two, as A, B, may be void, once in the portion subtracted for A and once for B. We must therefore add this number, which is $\frac{1}{2}r(r-1)(r-2)^n$. We have now subtracted the number of ways in which A, B, C may be void three times, and added it three times (in BC, CA, AB); we must therefore subtract it, and so on, exactly as in finding the number of numbers less than N and prime to it. The whole number of ways in which the n things may be distributed so that every body has at least one is then

$$r^n - r(r-1)^n + \frac{r(r-1)}{\underline{|2}}(r-2)^n - \ldots = \text{coefficient of } x^n \text{ in } (\epsilon^x - 1)^r \, \underline{|n}.$$

[See Wolstenholme's *Book of Mathematical Problems*, Exs. 163 and 239. Mr. Wolstenholme remarks that he gave these results to Mr. C. B. Clarke, to set in Queen's College, Cambridge, in either 1865 or 1866, before the appearance of Mr. Todhunter's *History of the Theory of Probability*, in which he discovered they had been known more than a century.]

3 and 4. The result in (3) is clearly the number of solutions in integers of the equation $x_1 + x_2 + \ldots + x_r = n$, zero allowed, and is therefore the coefficient of x^n in the expansion of $(1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots) \ldots$ to r factors; or in the expansion of $(1-x)^{-r}$. So if zero values be not allowed, the result required is the coefficient of x^n in $x^r(1-x)^{-r}$.

These results are $\dfrac{\underline{|r+n-1}}{\underline{|r-1} \, . \, \underline{|n}}$, and $\dfrac{\underline{|n-1}}{\underline{|r-1} \, . \, \underline{|n-r}}$.

[The PROPOSER remarks that Professor CAYLEY'S theorem (Quest. 2309) follows immediately as a corollary from (2). He also gives the following proof of the fourth part of the question:—

4. Suppose the n things all alike be placed in a row; we may then place amongst them $r-1$ points of partition and assign the r parts in order to the r different persons in order. The number of ways in which it can be done is therefore the number of ways in which $r-1$ points of partition may be assigned out of $n-1$ intervals. It is therefore the number of combinations of $(n-1)$ things $(r-1)$ at a time.]

2591. (Proposed by Professor WHITWORTH.) — Let AA_1, BB_1, CC_1 be the bisectors of the triangle ABC, intersecting in G_1; let AA_2, BB_2, CC_2 be the perpendiculars from the angular points on the opposite sides, intersecting in G_2; and let B_1C_1, B_2C_2 intersect in P; C_1A_1, C_2A_2 in Q; and A_1B_1, A_2B_2 in R. Then AP, BQ, CR will all be perpendicular to G_1G_2; and the triangle PQR will "circumscribe" the triangle ABC. Let A_i, B_i, C_i be the feet of the interior bisectors of the angles A, B, C, and A_e, B_e, C_e the feet of the exterior bisectors: then the six straight lines PA_i, QB_i, RC_i, PA_e, QB_e, RC_e pass three and three through four points which are the points of contact of the nine-point circle with the inscribed and escribed circles.

Solution by STEPHEN WATSON.

The equations of the bisectors and perpendiculars being comprehended in

$$\sin A \cdot \alpha = \sin B \cdot \beta = \sin C \cdot \gamma, \quad \cos A \cdot \alpha = \cos B \cdot \beta = \cos C \cdot \gamma,$$

the equation of the line G_1G_2 is easily found to be

$$\sin 2A \sin (B-C) \cdot \alpha + \sin 2B \sin (C-A) \cdot \beta + \sin 2C \sin (A-B) \cdot \gamma = 0 \ldots (1);$$

also the points P, Q, R are determined by the equations

$$-\frac{\alpha}{\sin (B-C)} = \frac{\beta}{\sin (C-A)} = \frac{\gamma}{\sin (A-B)} \quad \ldots\ldots\ldots\ldots (2),$$

$$\frac{\alpha}{\sin (B-C)} = -\frac{\beta}{\sin (C-A)} = \frac{\gamma}{\sin (A-B)} \quad \ldots\ldots\ldots\ldots (3),$$

$$\frac{\alpha}{\sin (B-C)} = \frac{\beta}{\sin (C-A)} = -\frac{\gamma}{\sin (A-B)} \quad \ldots\ldots\ldots\ldots (4).$$

For the equations of B_1C_1, B_2C_2 are

$$\sin A \cdot \alpha - \sin B \cdot \beta - \sin C \cdot \gamma = 0, \quad \cos A \cdot \alpha - \cos B \cdot \beta - \cos C \cdot \gamma = 0,$$

from which eliminating successively α, β, γ, we get the equalities in (2), and similarly (3) and (4) may be proved.

Take (1); first with the equation of AP in (2); next with that of BQ in (3); and lastly with that of CR in (4); in each case, it is easily shown, the lines are perpendicular; hence AP, BQ, CR are all perpendicular to G_1G_2.

From (2), (3), (4) it is plain each of the triads of points A, Q, R; B, R, P; C, P, Q lie in a straight line; hence the triangle PQR circumscribes the triangle ABC.

Again, from (2), (3), (4) and the well-known equations of the interior and exterior bisectors of the angles, we find the following equations:

$$PA_s \ldots . \sin \tfrac{1}{2}(B+C) \cdot \alpha - \sin \tfrac{1}{2}(B-C)(\beta-\gamma),$$

$$QB_s \ldots . \sin \tfrac{1}{2}(C+A) \cdot \beta - \sin \tfrac{1}{2}(C-A)(\gamma-\alpha),$$

$$RC_s \ldots . \sin \tfrac{1}{2}(A+B) \cdot \gamma - \sin \tfrac{1}{2}(A-B)(\alpha-\beta),$$

$$PA_e \ldots . \cos \tfrac{1}{2}(B+C) \cdot \alpha + \cos \tfrac{1}{2}(B-C)(\beta+\gamma),$$

$$QB_e \ldots . \cos \tfrac{1}{2}(C+A) \cdot \beta + \cos \tfrac{1}{2}(C-A)(\gamma+\alpha),$$

$$RC_e \ldots . \cos \tfrac{1}{2}(A+B) \cdot \gamma + \cos \tfrac{1}{2}(A-B)(\alpha+\beta),$$

and from these equations we find that the triads of lines

$$(PA_s, QB_s, RC_s), (PA_s, QB_e, RC_e), (QB_s, RC_e, PA_e), (RC_s, PA_e, QB_e)$$

pass respectively through the points determined by

$$\alpha : \beta : \gamma = \sin^2 \tfrac{1}{2}(B-C) : \sin^2 \tfrac{1}{2}(C-A) : \sin^2 \tfrac{1}{2}(A-B) \ldots (5),$$

$$\alpha : \beta : \gamma = -\sin^2 \tfrac{1}{2}(B-C) : \cos^2 \tfrac{1}{2}(C-A) : \cos^2 \tfrac{1}{2}(A-B) \ldots (6),$$

$$\alpha : \beta : \gamma = \cos^2 \tfrac{1}{2}(B-C) : -\sin^2 \tfrac{1}{2}(C-A) : \cos^2 \tfrac{1}{2}(A-B) \ldots (7),$$

$$\alpha : \beta : \gamma = \cos^2 \tfrac{1}{2}(B-C) : \cos^2 \tfrac{1}{2}(C-A) : -\sin^2 \tfrac{1}{2}(A-B) \ldots (8).$$

Now, in the *Messenger of Mathematics*, Vol. IV., p. 63, it is shown that (5) is the point of contact of the inscribed and nine-point circles, and in a similar manner it may be shown that (6), (7), (8) are the points of contact of the escribed circles with the same circle.

NOTE ON PRIME NUMBERS: by A. DE MORGAN.

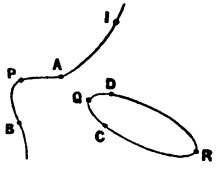
The following matter is slight, but curious, as showing the possibility of proving a proposition in the theory of numbers by a method which extends only to an assignable integer, and then fails.

It is said, but I believe not proved, that an even number is always the sum of two prime numbers. This, I suppose, was suggested by trial. Now if the proposition be *false* in the case of $2n$, we must have $2n -$ (any less odd prime) = (some less odd composite); and different primes require different composites. Therefore the number of less odd composites must be at least equal to the number of less odd primes. But this does not happen until $2n = 120$; therefore, so far, the theorem must be true.

2697. (Proposed by Professor SYLVESTER.)—Let I be a point of inflexion on a cubic. From I draw tangents meeting the cubic in P, Q, R; and from P draw tangents meeting it in A, B, C, D. Prove that the three opposite pairs of sides of the tetragon ABCD will respectively intersect in I, Q, R.

Solution by F. D. THOMSON, M.A.

I is the point of inflexion, P, Q, R the points of contact of the three tangents from it, A, B, C, D the points of contact of the four tangents from P.



Then since III is a straight line, the satellites PQR are in a straight line.

Again, since PAA, and also PBB, are in a straight line, therefore the satellites of PP, AB, AB are in a straight line. But I is the satellite of PP or P; therefore if x be the satellite of AB, I is the satellite of x. But the only points of which I is the satellite are I, P, Q, R; therefore x must coincide with I, Q, or R.

Similarly, each of the six chords joining A, B, C, D passes through either I, Q, or R. If, for example, AB passes through I, BC through R, and therefore BD through Q, then the satellites of AC, BB, IR are in a straight line; but PQR is a straight line, hence AC passes through Q.

Hence BD, CA meet in Q; AD, BC in R; and BA, CD in I.

2694. (Proposed by G. O. HANLON.)—A rope without weight passes over a smooth pulley, having a monkey at one end and a weight equal to the monkey at the other. The monkey suddenly runs up the rope to a height h with a uniform velocity; what is the resulting motion of the weight?

Solution by SEPTIMUS TEBAY, B.A.

Let P be the mass of the monkey, Q the mass of the weight at the other end of the rope, x the space through which Q has ascended at the time t, and T the tension of the rope. Then if the monkey begins to ascend with a velocity a, its position at the time t will be $at - x$.

Now the accelerating forces on P, Q are $\frac{T-Pg}{P}$, $\frac{T-Qg}{Q}$.

Therefore $\frac{d^2}{dt^2}(at-x) = \frac{T-Pg}{P}$, $\frac{d^2x}{dt^2} = \frac{T-Qg}{Q}$;

and eliminating T, $Q\frac{d^2x}{dt^2} - P\frac{d^2}{dt^2}(at-x) = (P-Q)g$.

Therefore $Q\frac{dx}{dt} - P\frac{d}{dt}(at-x) = (P-Q)gt + C$,

or $(P+Q)\frac{dx}{dt} = (P-Q)gt + Pa + C$.

To determine C, we must conceive the weight P to receive a sudden impulse Pa, which will produce an initial velocity in $Q = \frac{Pa}{P+Q}$. Therefore $C = 0$; and $x = \frac{P-Q}{P+Q}\cdot\frac{g}{2}t^2 + \frac{Pat}{P+Q}$.

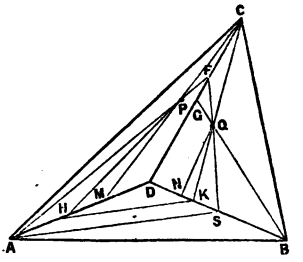
When $P = Q$, $x = \frac{1}{2}at$. Also, $h = at - x$.

2614. (Proposed by Capt. CLARKE, R.E., F.R.S.) — Three points are taken at random, one on each of three faces of a tetrahedron, show that the chance that the plane passing through them cuts the fourth face is $\frac{3}{4}$.

Solution by the PROPOSER.

Let D be the vertex of the tetrahedron, P, Q any points in the faces ADC, BDC. Produce AP, BQ to F and G, and suppose DG less than DF. Produce CP, CQ to meet AD, BD in H and K; let FQ produced meet DB in S; and join AS. Draw PM, QN parallel to CD. If the third point R be taken within the quadrilateral HASK, it is clear that the plane PQR will not cut the face ABC. Now the chance that R falls in this space is

$$\frac{AD.DS}{AD.DB} - \frac{HD.DK}{AD.DB} \quad \ldots\ldots (a).$$



Let DA$=f$, DB$=g$, DC$=h$, DM$=fx$, MP$=hy$, DN$=gx'$, NQ$=hy'$, DF$=nh$; HD$=\mu f$, DK$=\nu g$, DS$=\lambda g$; then we have

$$x + \frac{y}{n} = 1, \quad \frac{x}{\mu} + y = 1, \quad \frac{x'}{\nu} + y' = 1, \quad \frac{x'}{\lambda} + \frac{y'}{n} = 1;$$

therefore $n = \frac{y}{1-x}$, $\lambda = \frac{x'}{1-\frac{y'}{n}}$, $\mu = \frac{x}{1-y}$, $\nu = \frac{x'}{1-y'}$.

Substituting in (a), it becomes $\dfrac{x}{1-\dfrac{y'}{n}} - \dfrac{xx'}{(1-y)(1-y')}$.

The equation to the line FB is $x' + \dfrac{y'}{n} = 1$, so that, if we integrate with respect to x', y', x, y, successively, the probability that the plane PQR does not cut ABC is

$$4\int_0^1\int_0^{1-y}\int_0^n\int_0^{1-\frac{y'}{n}}\left[\frac{nx'}{n-y'} - \frac{xx'}{(1-y)(1-y')}\right]dy\,dx\,dy'\,dx'$$

$$= 2\int_0^1\int_0^{1-y}\int_0^n\left[1 - \frac{1-x}{y}y' - \frac{x}{1-y}\cdot\frac{\left(1-\frac{1-x}{y}y'\right)^2}{1-y'}\right]dy\,dx\,dy'$$

$$= \int_0^1\int_0^{1-y}\left[\frac{y}{1-x} + 2\frac{x(1-x)}{y(1-y)} - \frac{3x}{1-y} + \frac{2x}{1-y}\cdot\frac{(1-x-y)^2}{y^2}\log\frac{1-x-y}{1-x}\right] \times dy\,dx.$$

Let F, G, H, K be the values of the four separate terms in this expression, then $F = \frac{1}{4}$; $G = 2\int_0^1\left[\frac{1}{6} + \frac{1}{6y} - \frac{1}{8}y\right]dy = \frac{1}{8}\left[\log y\right]_0^1$; $H = -\frac{3}{4}$;

$$K = 2\int_0^1\int_0^{1-y}\frac{1}{y^2}\left[x(1-y) - 2x^2 + \frac{x^3}{1-y}\right]\log\frac{1-x-y}{1-x}\,dy\,dx$$

$$= \frac{1}{6}\int_0^1\left[\left(y + \frac{3}{1-y} - 3\right)\log y + \left(\frac{1}{y^2} - \frac{3}{y} + 3 - y\right)\log(1-y) + \frac{9}{2} - \frac{1}{y} - \frac{y}{2}\right]dy$$

$$= \left[-\frac{1}{2}\log y\log(1-y) + \left(\frac{1-y^2}{12} - \frac{1-y}{2}\right)\log(1-y) + \left(\frac{y^2}{12} - \frac{y}{2}\right)\log y\right.$$

$$\left. + \frac{1}{12}y - \frac{1-y}{6y}\log(1-y) - \frac{1}{3}\log y + \frac{3}{4}y - \frac{1}{24}y^2 \ldots\right]_0^1 = \frac{5}{8} - \frac{1}{8}\left[\log y\right]_0^1;$$

therefore $$F + G + H + K = \tfrac{1}{2}.$$

This result is obtained subject to the condition that DG is less than DF. It is clear that, for the case in which DG is greater than DF, we should obtain also $\frac{1}{2}$. Therefore the chance that the plane through PQR does not cut the base is $\frac{1}{2}$; that it *does*, the chance is $\frac{3}{4}$.

2604. (Proposed by J. J. WALKER, M.A.)—Prove that in the third system of pulleys (these being supposed n in number, of equal weight w, and equal diameter $2a$), the distance of the centre of the parallel tensions of the cords from the first cord is equal to

$$\frac{4\left\{(n-1)2^{n-1}+1\right\}W + n\left\{(n-3)2^n + n + 3\right\}w}{2(2^n-1)W}\,a.$$

Solution by the PROPOSER.

If P is the power which, applied to the free end of the first cord, will just support the weight W, and $T_1, T_2, \ldots T_n$ represent the tensions of the successive cords, we have

$$P = T_1,\ 2T_1 = T_2 - w,\ 2T_2 = T_3 - w, \ldots . 2T_{n-1} = T_n - w,\ T_1 + T_2 \ldots + T_n = W.$$

Multiplying these equations by $2^n - 1, 2^{n-1} - 1, \ldots . 2 - 1, 1$, respectively, and adding all together, $T_1, T_2, \ldots . T_n$ are eliminated, and we have

$$(2^n - 1)\,P = W - (2^n - 1 - n)\,w, \text{ or } P = \frac{W - (2^n - 1 - n)\,w}{2^n - 1}.$$

$$\text{Again } T_1 = P,\ T_2 = 2T_1 + w = 2P + w = \frac{2W - (2^n - 1 - 2n)\,w}{2^n - 1},$$

$$T = 2T_2 + w = \frac{2^2 W - (2^n - 1 - 2^2 n)\,w}{2^n - 1} \ldots .$$

$$T_n = 2T_{n-1} + w = \frac{2^{n-1} W - (2^n - 1 - 2^{n-1} n)\,w}{2^n - 1}.$$

Now, if x be the required distance,

$$\frac{xW}{a} = T_2 + 2T_3 + 3T_4 + \ldots . + (n-1)\,T_n,$$

$$\text{or } \frac{x(2^n - 1)\,W}{a} = (1.2 + 2.2^2 + 3.2^3 + \ldots . + \overline{n-1}\,2^{n-1})\,(W + nw)$$

$$- (1 + 2 + 3 \ldots . + n - 1)\,(2^n - 1)\,w$$

$$= \left\{(n-1)\,2^n + 2\right\}(W + nw) - \tfrac{1}{2} n (n-1)(2^n - 1)\,w$$

$$= 2\left\{(n-1)\,2^{n-1} + 1\right\} W + \tfrac{1}{2}\left\{(n-3)\,2^n + n + 3\right\} nw.$$

2486. (Proposed by G. O. HANLON.)—Find a point on an ellipse such that the normal produced to meet the curve again may cut off a maximum area.

Solution by M. JENKINS, B.A.; *the* PROPOSER; *and others.*

By taking consecutive normals we see that the chord formed by the normal produced must be bisected at its stationary point. Hence the osculating circle must cut the chord again in the same point as the ellipse. Therefore the chord, being a common chord of the ellipse and osculating circle, is equally inclined to the axis with the tangent (corresponding to the normal).

Hence the tangent and normal at the required point must be equally inclined to the axis; and we can thus easily show that the tangent of the inclination to the major axis of the radius vector to the required point is equal to the square of the ratio of the minor to the major axis.

Let CA, CB be the semi-major and semi-minor axes respectively of the ellipse; CU a radius vector of the auxiliary circle such that $\tan \mathrm{CUA} = \frac{\mathrm{BC}}{\mathrm{AC}}$. Then, if UPN be an ordinate cutting the ellipse in P, P is one of the required points.

We shall also have the following properties. Let POH be the normal chord; O the middle and stationary point of the chord. Then POH is the tangent at O to the evolute of the ellipse. But CO and CP are conjugate respectively to the normal and tangent at P. Hence CO and CP are also equally inclined to the axes; and therefore CP cuts the evolute in a point such that the tangent at it is conjugate to CP and cuts off a maximum area.

[Under a slightly different form of enunciation, the problem is to find in an ellipse a point such that, if there be drawn through it a diameter and a normal chord, the area contained by these lines and the curve shall be a maximum. Putting a and b for the semi-axes, $c^2 = a^2 - b^2$, and θ for the inclination of the required normal to the major axis, the area in question is

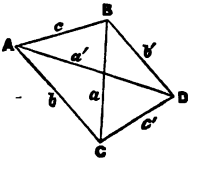
$$\frac{2a^2b^2c^2 \sin 2\theta}{c^4 \sin^2 2\theta + 4a^2b^2} + ab \tan^{-1}\left(\frac{c^2 \sin 2\theta}{2ab}\right),$$

which is readily found to have its maximum value $\frac{2a^2b^2c^2}{(a^2+b^2)^2} + ab \tan^{-1}\left(\frac{c^2}{2ab}\right)$ when $\theta = \frac{1}{4}\pi$; the position of the normal thus determined being the same as that found in Mr. JENKINS'S solution.]

2690. (Proposed by M. COLLINS, B.A.)—If four points on a plane be the extremities of the two straight lines a, a', or of the pair b, b', or of the pair c, c'; then, a, b, c being the sides of a triangle, prove that

$$(a^2a'^2 + b^2b'^2 + c^2c'^2)(a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)$$
$$= 2(a^4a'^2 + a'^4a^2 + b^4b'^2 + b'^4b^2 + c^4c'^2 + c'^4c^2) + a^2b^2c^2 + a^2b'^2c'^2 + a'^2b^2c'^2 + a'^2b'^2c^2.$$

Solution by S. BILLS; J. WILSON; *and others.*



Let A, B, C, D be the four points; then, since the angle ACD = ACB + DCB, we shall have the well-known relation

$$1 - \cos^2 \mathrm{ACB} - \cos^2 \mathrm{DCB} - \cos^2 \mathrm{ACD} + 2 \cos \mathrm{ACB} \cos \mathrm{DCB} \cos \mathrm{ACD} = 0 \ldots (1).$$

But $\cos \mathrm{ACB} = \frac{a^2 + b^2 - c^2}{2ab}$, $\cos \mathrm{DCB} = \frac{a^2 + c'^2 - b'^2}{2ac'}$,

and $\cos \mathrm{ACD} = \frac{b^2 + c'^2 - a'^2}{2bc'}$.

Substituting these results in (1), we readily find

$$4a^2b^2c'^2 - c'^2(a^2 + b^2 - c^2)^2 - b^2(a^2 + c'^2 - b'^2)^2 - a^2(b^2 + c'^2 - a'^2)^2 + (a^2 + b^2 - c^2)(a^2 + c'^2 - b'^2)(b^2 + c'^2 - a'^2) = 0 \ldots . (2).$$

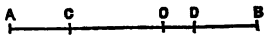
This equation (2) is equivalent to the one in the question, and may be easily reduced to the same form.

2695. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If a given finite straight line of length a be divided at random in two points, the chance that the product of the three parts shall exceed $\frac{1}{108}a^3$ is

$$\frac{4}{9}\int_{\frac{4}{9}\pi}^{\frac{8}{9}\pi} \sin\tfrac{1}{2}\theta\,(1+2\cos 3\theta)^{\frac{1}{2}}\,d\theta.$$

Solution by STEPHEN WATSON.

Let C and D be the points in the given line AB, and bisect CB in O. Put $AC=ax$, $OD=ay$; then by the question



$$x\left\{\tfrac{1}{4}(1-x)^2-y^2\right\} > \frac{1}{108}, \quad \text{therefore} \quad y < \pm\tfrac{1}{2}\left\{(1-x)^2-\frac{1}{27x}\right\}^{\frac{1}{2}};$$

hence the number of values y takes while x remains fixed is $\left\{(1-x)^2-\frac{1}{27x}\right\}^{\frac{1}{2}}$; therefore, doubling because C and D may be interchanged, the required chance is

$$2\int\left\{(1-x)^2-\frac{1}{27x}\right\}^{\frac{1}{2}}dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the limits being derived from $(1-x)^2x=\frac{1}{27}$, in which, if we put $\frac{2}{3}(1+\cos\theta)$ for x, we have $4\cos^3\theta-3\cos\theta=-\frac{1}{2}$, or $\cos 3\theta=-\frac{1}{2}$, whence $\theta=\frac{2}{9}\pi$, or $\frac{4}{9}\pi$, or $\frac{8}{9}\pi$. The first of these values of θ gives $x>1$, and is therefore inadmissible; the second gives the greater value of x, and the third the less. Putting, then, $x=\frac{2}{3}(1+\cos\theta)$ in (1), it easily gives the probability stated in the question.

2687. (Proposed by W. S. M'CAY, B.A.)—If normals to circumscribing conics at the vertices of a triangle meet in a point, and if also normals to inscribed conics at the points of contact meet in a point; show that the locus of this point is the same in both cases.

Solution by STEPHEN WATSON.

If the equation of the circumscribing conic be $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$, the tangents at the angles are

$$n\beta+m\gamma=0, \quad l\gamma+n\alpha=0, \quad \text{and} \quad m\alpha+l\beta=0;$$

and the normals, being respectively perpendicular to those, are easily found to be

$$\left.\begin{aligned} n\,(\gamma+\beta\cos A) &= m\,(\beta+\gamma\cos A)\\ l\,(\alpha+\gamma\cos B) &= n\,(\gamma+\alpha\cos B)\\ m\,(\beta+\alpha\cos C) &= l\,(\alpha+\beta\cos C)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, let the inscribed conic be $(l'\alpha)^{\frac{1}{2}}+(m'\beta)^{\frac{1}{2}}+(n'\gamma)^{\frac{1}{2}}=0$, then the normals at the points of contact are

$$\left.\begin{aligned} n'(\gamma + \alpha\cos B) &= m'(\beta + \alpha\cos C) \\ l'(\alpha + \beta\cos C) &= n'(\gamma + \beta\cos A) \\ m'(\beta + \gamma\cos A) &= l'(\alpha + \gamma\cos B) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2):$$

for the first must be parallel to $\gamma + \alpha\cos B = 0$, and pass through the point $(\alpha = 0,\ m'\beta = n'\gamma)$, hence its equation is found to be as above. Similarly for the others.

Eliminating (l, m, n) from (1), and (l', m', n') from (2), we obtain in both cases the same result as the equation of the required locus, viz.,

$$(\cos A - \cos B\cos C)(\beta^2 - \gamma^2)\alpha + (\cos B - \cos C\cos A)(\gamma^2 - \alpha^2)\beta + (\cos C - \cos A\cos B)(\alpha^2 - \beta^2)\gamma = 0 \quad \ldots. (3).$$

Put successively $\alpha, \beta, \gamma = 0$; then the results give

$$\frac{\alpha}{\cos A - \cos B\cos C} = \frac{\beta}{\cos B - \cos C\cos A} = \frac{\gamma}{\cos C - \cos A\cos B} \quad \ldots (4);$$

hence (3) meets the sides of the triangle of reference in three points such that, when joined to the opposite angles, the three joining lines are concurrent.

2627. (Generalized from WHITWORTH'S *Choice and Chance*, page 65.)— How many different signals can be made with a set of n different flags, using r of them at a time, and arranging them on s masts where $n > r > s$?

Solution by PROFESSOR WHITWORTH.

The r flags can be selected and arranged in one row in $\frac{\lfloor n}{\lfloor n-r}$ different ways. And then $s-1$ points of partition can be placed amongst them in $\frac{\lfloor r-1}{\lfloor r-s \lfloor s-1}$ different ways. Hence the number of signals will be

$$\frac{\lfloor n \lfloor r-1}{\lfloor n-r \lfloor r-s \lfloor s-1}.$$

If it be permitted to leave some of the masts vacant, then by Question 2414 (3) the $s-1$ points of partition can be placed in $\frac{\lfloor s+r-1}{\lfloor r \lfloor s-1}$ different ways and the number of signals will therefore be

$$\frac{\lfloor n \lfloor s+r-1}{\lfloor n-r \lfloor r \lfloor s-1}.$$

2506. (Proposed by the Rev. R. TOWNSEND, F.R.S.)— Determine the two positions of the line whose chord of intersection with an assigned one of three given circles shall be cut harmonically by the other two.

I. *Solution by the* PROPOSER.

Let A, B, C be the three circles, and P, Q, R their three centres; then, since a chord XY of C cut harmonically by A and B, being the diameter of a circle orthogonal to A and B, must pass through a determinable point O on the line PQ and have its middle point Z on the radical axis L of A and B, therefore the circle on OR as diameter will intersect L at the two middle points Z_1 and Z_2 of the two required chords X_1Y_1 and X_2Y_2 through O, and therefore, &c.

II. *Solution by the* REV. J. WOLSTENHOLME, M.A.

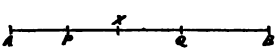
Let the assigned circle be A, the two others B, C. A straight line divided harmonically by any conics P, Q envelopes a conic R, which touches the eight tangents to P, Q at the common points. For (assuming the envelope to be a conic) let α be a common tangent to R and P, then α touches P and is divided harmonically by P, Q; hence one of the points where it meets Q must coincide with its point of contact with P, or α touches P at one of the common points of P, Q. Now for two circles A, B, the envelope R will touch the tangents drawn to these circles at their common points; and hence, from the points at infinity, we see that the centres of the circles are the foci of the conic. If then A intersects B, C in real points at a finite distance, and a, b, c be the centres of the circles, draw a conic with foci a, b touching one (and therefore all) of the tangents to A, B at their common points; and a conic with foci a, c touching the tangents to A, C at their common points: the common tangents of these two will be the lines required. Since the conics have one common focus, they will have only two common tangents, besides the impossible two from the circular points. If the circles do not intersect, the straight lines divided harmonically must intersect both circles in impossible points; but it would be perfectly easy to find the equations.

2605. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—A horizontal beam AB, supported but not fixed at both ends A and B, is traversed from end to end by a moving load W distributed uniformly over a segment of it of constant length PQ. Show that the greatest bending moment M at any point X of the beam corresponds to the position of the load in its passage over the point for which PX : QX = AX : BX, and has for value

$$M = \frac{W}{AB^2} \cdot AX \cdot BX \cdot (AB - \tfrac{1}{2}PQ).$$

I. *Solution by* J. J. WALKER, M.A.

Let AX = a, BX = b, AB = $a + b = L$, PQ = l, x = distance of any point between P and X from A, AP = x', y = distance of any point between Q and X from B, BQ = y'. Then the bending moment at X will be



$$M = \frac{W}{lL}\left(\int_{x'}^{a} bx\,dx + \int_{y'}^{b} ay\,dy\right) = \frac{W}{2lL}(abL - bx'^2 - ay'^2).$$

This will be a maximum when $bx\,dx' + ay\,dy' = 0$.
But $x' + y' = L - l$, whence $dx' + dy' = 0$. The condition for the maximum becomes, therefore, $bx = ay'$, or $a : x' = b : y'$, or $a : a - x' = b : b - y'$, or $AX : PX = BX : QX$, which proves the first part of the question.

Again, from the equations $bx' - ay' = 0$, $x' + y' = L - l$, we have

$$x = \frac{a}{L}(L-l), \quad y' = \frac{b}{L}(L-l),$$

and substituting these values in the general expression for M, the maximum moment becomes

$$\frac{W}{2lL}\,ab\left\{L - \frac{(L-l)^2}{L}\right\} = \frac{W}{L^2}\,ab\,(L - \tfrac{1}{2}l) \text{ or } \frac{W}{AB^2}\,AX \,.\, BX\,(AB - \tfrac{1}{2}PQ).$$

II. *Solution by the* PROPOSER.

By known principles (see *Reprint*, Vol. IV., p. 29), if PQ lie altogether between A and X, then

$$M = \frac{W}{AB} \cdot \tfrac{1}{2}(AP + AQ) \,.\, BX \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

if it lie altogether between B and X, then

$$M = \frac{W}{AB} \cdot \tfrac{1}{2}(BP + BQ) \,.\, AX \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and if it lie partly between A and X, and partly between B and X, then

$$M = \frac{W}{AB} \cdot \left[\frac{PX}{PQ} \cdot (AX - \tfrac{1}{2}PX) \,.\, BX + \frac{QX}{PQ} \cdot (BX - \tfrac{1}{2}QX) \,.\, AX\right] \ldots\ldots (3),$$

which, since in that case $PX + QX = PQ$, becomes by reduction

$$M = \frac{W}{AB} \cdot \left[AX \,.\, BX - \frac{1}{2\,.\,PQ} \cdot (BX \,.\, PX^2 + AX \,.\, QX^2)\right] \ldots\ldots (4).$$

The maxima values of M for these three cases, respectively, correspond, of course, for the first to the maximum value of $\frac{1}{2}(AP + AQ)$, that is to $AX - \frac{1}{2}PQ$, since then Q coincides with X; for the second to the maximum value of $\frac{1}{2}(BP + BQ)$, that is to $BX - \frac{1}{2}PQ$, since then P coincides with X; and for the third to the minimum value of $BX \,.\, PX^2 + AX \,.\, QX^2$, that is to $\frac{PQ^2}{AB} \,.\, AX \,.\, BX$, since then $BX \,.\, PX = AX \,.\, QX$, or, which is the same thing, $PX : QX = AX : BX$; thus giving, for the maxima values of M in the three cases respectively,

$$M = \frac{W}{AB} \,.\, AX \,.\, BX \,.\, \left(1 - \frac{1}{2}\frac{PQ}{AX}\right), \text{ for the first case;}$$

$$M = \frac{W}{AB} \,.\, AX \,.\, BX \,.\, \left(1 - \frac{1}{2}\frac{PQ}{BX}\right), \text{ for the second case;}$$

$$M = \frac{W}{AB} \,.\, AX \,.\, BX \,.\, \left(1 - \frac{1}{2}\frac{PQ}{AB}\right), \text{ for the third case,}$$

which last, since $AX + BX = AB$, is, as above stated, the greatest of the three, and has evidently the value there given.

N.B.—For the two extreme cases in which PQ = 0 and = AB, the last formula gives, as of course it ought, the two well known values of M for those cases respectively, viz., $M = \frac{W}{AB} \cdot AX \cdot BX$, and $= \frac{1}{2} \frac{W}{AB} \cdot AX \cdot BX$. (See *Reprint*, Vol. IV., p. 30.)

2453. (Proposed by Professor WHITWORTH.)—The total number of partitions of n different things (*i.e.* the number of ways in which n different things can be distributed into 1, 2, 3, ... or n indifferent parcels) is the coefficient of x^n in the expansion of $\lfloor n \cdot \epsilon^{e^x - 1}$

Solution by the REV. J. WOLSTENHOLME, M.A.

By Quest. 2414 the number of ways of distributing n different things to r different people so that each may have at least one, is the coefficient of x^n in the expansion of $\lfloor n \, (e^x + 1)^r$. Hence the number of ways of dividing the n things into r heaps is $(\lfloor r)^{-1}$ multiplied by the former number, since permuting the heaps gives no different partition. The whole number of partitions is therefore $\lfloor n$ times the coefficient of x^n in the expansions of

$$(\epsilon^x - 1)^1 + \frac{1}{\lfloor 2}(\epsilon^x - 1)^2 + \ldots + \frac{1}{\lfloor n}(\epsilon^x - 1)^n,$$

and since the coefficients of x^n in subsequent terms are zero, we may consider this series to go on to infinity. The whole number is then the coefficient of x^n in the expansion of $\lfloor n \cdot \epsilon^{e^x - 1}$.

2639. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—The direction angles of three lines through the origin are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$; required the equation of the cone of revolution of which they are sides.

Solution by J. J. WALKER, M.A.

Let (l, m, n), (l', m', n'), (l'', m'', n'') be the direction-cosines of the three given lines. The equation to a plane cutting them at points equidistant (by r) from the origin will be

$$\frac{L}{D} x + \frac{M}{D} y + \frac{N}{D} z = r,$$

where $D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ l & m & n & 1 \\ l' & m' & n & 1 \\ l'' & m'' & n'' & 1 \end{vmatrix}$, and L, M, N are the minors formed by omitting the second, third, and fourth rows and columns successively.

At the common section of this plane and the sphere

$$x^2 + y^2 + z^2 = r,$$

$$(L^2 - D^2)x^2 + (M^2 - D^2)y^2 + (N^2 - D^2)z^2 + 2MNyz + 2LNxz + 2LMxy = 0.$$

This is the equation to a cone having its vertex at the origin, passing through the three given lines, and having its sections by planes cutting these sides in points equidistant from the vertex circular. Hence it is the required cone.

2521. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Two lines in space intersect a plane conic; show that the ruled surface determined by the three as directors is a quadric.

Solution by A. W. PANTON, B.A.

Describe a quadric through the two given lines, the two points on the conic consecutive to the points where the lines meet it, and any third point on the conic. This quadric is cut by the plane of the given conic in a conic having in common with the former three points, and the tangents at two of them. The conics are therefore coincident, and we have a quadric passing through the given lines and the given conic.

2630. (Proposed by W. S. BURNSIDE, M.A.)—Show that the degree of the locus of the foci of a system of conics, subject to any four conditions, is three times as great as that of the locus of their centres.

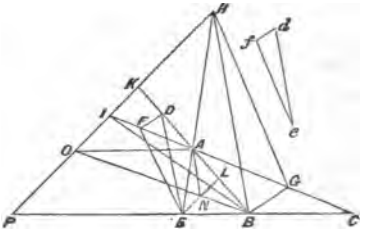
Solution by the PROPOSER.

Whenever a real focus goes off to infinity it is accompanied with the two imaginary foci, which take their place at the imaginary circular points, and the centre follows too, whence the line at infinity has three foci and one centre only *corresponding*, on it.

2607. (Proposed by W. HOPPS.)—In three straight lines given in position, to determine the positions of the vertices of a triangle of given or minimum magnitude, and of given species.

Solution by the PROPOSER.

Let A, B, C be the intersections of the three given lines in pairs, and suppose the vertices D, E, F of a triangle of the same magnitude and species as the given triangle *def*, to be posited in these lines as in the diagram. Draw lines from B respectively parallel to DE, DF, the former meeting the line through E, A at H, and the latter the line AC at G, and join G, H. Then, by parallels, we have



$$AH : AE = AB : AD = AG : AF;$$

hence the triangles BHG, DEF are similar. Suppose the circle passing through B, G, H to cut CA in I, and draw the lines BI, HI, the latter cutting BA in K. Now B, G, H, I being points in the circumference of a circle, the angles BIC, CIH are respectively equal to BHG, GBH, which are given angles; hence the point I and the position of the line HI are given. Let P be the intersection of HI, CB, and from A, E draw lines respectively parallel to DE, HP, to meet at N, the latter also meeting AB at L; and from B draw a line through N, meeting HP at O. Then we have

$$KB : LD = BH : DE = AH : AE = KA : LA,$$

$$\therefore\ KB : KA\ (1) = LD : LA = DE : AN\ (2) = LE : LN = KP : KO,$$

consequently the line joining O, A is parallel to BC; also from (1) and (2), AN is given in length. We have, therefore, the following

Construction.—Determine I the intersection with CA of the circular arc containing an angle equal to *def*, and whose chord is BC. Through I draw a line meeting CA at K, CB at P, and a line through A, parallel to BC, at O, and making with CI the angle KIC = *edf*. Draw OB, and from A inflect a line equal to a fourth proportional to KB, KA and *de*, to meet OB at N. Draw NE parallel to KP, and ED to AN; also make the angle DEF = *def* and join F, D. Then from the analysis, it is obvious that the triangle DEF, whose vertices are posited in the given lines, is of the same magnitude and species as the given triangle *def*.

Again, from (1) and (2) in the analysis, we have KB : KA = DE : AN. Whence it is evident that when DE, or the triangle DEF, is a minimum, AN is also a minimum; which is manifestly the case when AN is perpendicular to OB. Moreover, when the triangle DEF is *not* a minimum, it is clear that its vertices, collectively, may have two different positions in the given lines, inasmuch as the line AN may be inflected to meet OB at two different points. And finally, it may be remarked that, from the solution preceding, it may readily be shown that in three given lines the vertices of a triangle of given magnitude and species may have several different positions, and consequently each position has a corresponding triangle of minimum magnitude and of the same species; but, in the foregoing solution, it is not shown whether these minimum triangles are, or are not, of the same magnitude; however, when the given triangle is equiangular, it may be easily shown that there subsists but one minimum triangle of the same species.

[Another solution is given in Townsend's *Modern Geometry*, Vol. I., Art. 55.]

2482. (Proposed by H. R. GREER, B.A.)—A quadrilateral circumscribes an ellipse, the eccentric angles of the points of contact being α, β, γ, δ. Prove that the condition that a pair of opposite vertices should lie on a confocal conic may be thus expressed:—

$$a^2(\cos A+\cos B+\cos C+\cos D)(\cos A+\cos B-\cos C-\cos D)\times$$
$$(\cos C+\cos A-\cos B-\cos D)(\cos B+\cos C-\cos A-\cos D) =$$
$$b^2(\sin A+\sin B+\sin C+\sin D)(\sin A+\sin B-\sin C-\sin D)+$$
$$(\sin C+\sin A-\sin B-\sin D)(\sin B+\sin C-\sin A-\sin D);$$

where $A=\frac{1}{2}(-\alpha+\beta+\gamma+\delta)$, $B=\frac{1}{2}(\alpha-\beta+\gamma+\delta)$,
$C=\frac{1}{2}(\alpha+\beta-\gamma+\delta)$, $D=\frac{1}{2}(\alpha+\beta+\gamma-\delta)$.

Solution by the PROPOSER.

Let tangents be drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points whose eccentric angles are α and β; then the coordinates of the intersection of these are $x=a\frac{\cos\epsilon}{\cos\eta}$, $y=b\frac{\sin\epsilon}{\cos\eta}$, where $\epsilon=\frac{1}{2}(\alpha+\beta)$ and $\eta=\frac{1}{2}(\alpha-\beta)$. The condition that this point should lie upon the confocal ellipse $\frac{x^2}{a^2+h^2}+\frac{y^2}{b^2+h^2}=1$ is

$$a^2(b^2+h^2)\cos^2\epsilon+b^2(a^2+h^2)\sin^2\epsilon=(a^2+h^2)(b^2+h^2)\cos^2\eta.$$

Similarly, the condition that the intersection of tangents at the points defined by the eccentric angles γ and δ should lie on the same confocal is, putting $\upsilon=\frac{1}{2}(\gamma+\delta)$ and $\omega=\frac{1}{2}(\gamma-\delta)$,

$$a^2(b^2+h^2)\cos^2\upsilon+b^2(a^2+h^2)\sin^2\upsilon=(a^2+h^2)(b^2+h^2)\cos^2\omega.$$

Let us eliminate h^2 from these two equations. Multiplying crossways, we get an equation which is equivalent to the following pair:

$$\left.\begin{aligned}a^2(b^2+h^2)&=\Lambda S\\ b^2(a^2+h^2)&=\Lambda K\end{aligned}\right\},\quad\text{where}\quad\left\{\begin{aligned}S&=\sin^2\upsilon\cos^2\eta-\sin^2\epsilon\cos^2\omega\\ K&=\cos^2\epsilon\cos^2\omega-\cos^2\eta\cos^2\upsilon\end{aligned}\right\};$$

after which we easily get the eliminant in the form

$$(S\cos^2\epsilon+K\sin^2\epsilon)(a^2K-b^2S)=(a^2-b^2)SK\cos^2\eta;$$

or, putting $T=\sin^2\upsilon\cos^2\epsilon-\cos^2\upsilon\sin^2\epsilon$, in the form

$$a^2K(T-S)=b^2S(T-K),\quad i.e.,\text{ in the form}$$

$$a^2(\cos\epsilon\cos\omega+\cos\eta\cos\upsilon)(\cos\epsilon\cos\omega-\cos\eta\cos\upsilon)$$
$$\times(\sin\upsilon\sin\eta+\sin\epsilon\sin\omega)(\sin\upsilon\sin\eta-\sin\epsilon\sin\omega)$$
$$=b^2(\sin\upsilon\cos\eta+\sin\epsilon\cos\omega)(\sin\upsilon\cos\eta-\sin\epsilon\cos\omega)$$
$$\times(\sin\omega\cos\epsilon+\sin\eta\cos\upsilon)(\sin\omega\cos\epsilon-\sin\eta\cos\upsilon),$$

observing that $2\cos\epsilon\cos\omega=\cos(\epsilon+\omega)+\cos(\epsilon-\omega)$, and applying analogous transformation to the other terms, we deduce the required result. In it α, β, γ, δ are interchangeable; which proves the theorem (Quest. 1819) that if a quadrilateral circumscribe an ellipse having one pair of its opposite vertices on a confocal conic, each other pair will also lie on a confocal conic.

2465. (Proposed by the late R. H. WRIGHT, M.A.)—If the tangent drawn to the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, at one end of a chord whose pole is (h, k), be represented by $a^2yy' + b^2xx' = a^2b^2$, find the equation to the tangent at the other end of the polar.

Solution by W. H. LAVERTY, B.A.

Let (x_1, y_1) be the other extremity of the polar, and let $b^2h^2 + a^2k^2 - a^2b^2 = z^4$; then, for the intersections of the polar of (h, k) with the curve, we have

$$x_1 = \frac{a^2}{b^2h^2 + a^2k^2}(b^2h + z^2k); \quad x' = \frac{a^2}{b^2h^2 + a^2k^2}(b^2h - z^2k);$$

$$y_1 = \frac{b^2}{b^2h^2 + a^2k^2}(a^2k - z^2h); \quad y' = \frac{b^2}{b^2h^2 + a^2k^2}(a^2k + z^2h);$$

therefore $x_1 = x' \cdot \dfrac{b^2h + z^2k}{b^2h - z^2k}$; $\qquad y_1 = y' \cdot \dfrac{a^2k - z^2h}{a^2k + z^2h}$.

Hence the equation to the tangent at (x_1, y_1) is $a^2yy_1 + b^2xx_1 = ab$, which, by substituting and multiplying out, becomes

$$(a^2k - z^2h)(b^2h - z^2k)\,a^2yy' + (a^2k + z^2h)(b^2h + z^2k)\,b^2xx' = (a^2k + z^2h)(b^2h - z^2k)\,a^2b^2.$$

2425. (Proposed by T. T. WILKINSON, F.R.A.S.)—A heavy cone stands inverted on its vertex, its axis making an angle α with the horizon; find the magnitude and direction of the least power which, when applied at the centre of its base, will just sustain the cone in that position.

Solution by J. J. WALKER, M.A.

Let P be the required force, Q the resultant of the pressures on the vertex of the cone, W its weight. These three forces will lie in the vertical plane passing through the axis of the cone. Resolving them horizontally and vertically, and taking the moments with respect to the vertex, the three conditions of equilibrium give, θ and ϕ being the angles which the directions of P and Q respectively make with the horizontal,

$$P\cos\theta + Q\cos\phi = 0 \ldots\ldots\ldots (1),$$

$$W + P\sin\theta + Q\sin\phi = 0 \ldots\ldots\ldots (2),$$

$$\tfrac{3}{4}W\cos\alpha + P\sin(\alpha + \theta) = 0 \ldots\ldots\ldots (3).$$

From (3), $P = -\dfrac{3}{4}\dfrac{\cos\alpha}{\sin(\alpha + \theta)}W$, and will therefore be least when $\theta = 90° - \alpha$, or P lies in the base of the cone, its value being then

$$P = -\tfrac{3}{4}\cos\alpha \,.\, W \ldots\ldots\ldots (4),$$

From (1) and (2), $\tan\phi = \dfrac{W + P\sin\theta}{P\cos\theta} = -\dfrac{1}{3}\dfrac{1+3\sin^2\alpha}{\sin\alpha\cos\alpha}$ or $-\dfrac{1}{3}(4\tan\alpha + \cot\alpha)$,

when P is the least force; whence $\cos\phi = \dfrac{-3\sin\alpha\cos\alpha}{\sqrt{(1+15\sin^2\alpha)}}$.

Finally, from (1), $Q = \dfrac{-P\cos\theta}{\cos\phi} = -\dfrac{W}{4}\sqrt{(1+15\sin^2\alpha)}$.

2382. (Proposed by H. R. GREER, B.A.)—Given a quartic curve Q, and a point on it P; suppose that from p there are drawn the ten tangents to Q, omitting that at P, touching Q in ten points (t) and meeting it again in ten points (r). The t points, we know, lie on a cubic K; prove that the r points lie on a quartic U, with a point of osculation at p, touching K at p (counting for four points of intersection), and in four other points.

Solution by JAMES DALE.

The equation which gives the points where the line meets the curve which joins the given point p (x_1, y_1, z_1) to a variable point (x, y, z), is

$$l^4Q + l^3m(\Delta_1Q) + \tfrac{1}{2}l^2m^2(\Delta_1^2Q) + lm^3(\Delta Q_1) + m^4Q_1 = 0,$$

where Q, Δ_1Q, Δ_1^2Q, ΔQ_1, Q_1 are respectively of the 4th, 3rd, 2nd, 1st, and 0th degree in x, y, z. If the point (x_1, y_1, z_1) lie on $Q=0$, then $Q_1=0$, and the equation becomes $l^3Q + l^2m\Delta_1 + \tfrac{1}{2}lm^2\Delta_1^2 + m^3\Delta = 0$

(writing Δ_1, Δ_1^2, Δ instead of Δ_1Q, Δ_1^2Q, ΔQ_1). The condition that this equation should have two equal roots is

$$\left\{27\Delta^2Q - 9\Delta\Delta_1\Delta_1^2 + \tfrac{1}{2}(\Delta_1^2)^3\right\}Q = \left\{\tfrac{1}{4}(\Delta_1^2)^2 - 4\Delta\Delta_1\right\}(\Delta_1)^2.$$

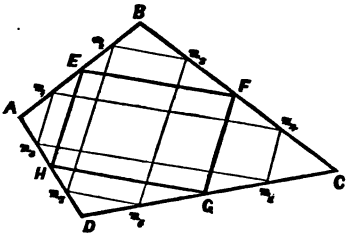
The form of the equation shows that it represents a locus of the 10th degree, touching Q in the points where Q meets Δ_1. The other points where Q meets the locus lie on the curve $(\Delta_1^2)^2 - 16\Delta\Delta_1 = 0$. Hence the points of contact of the ten tangents drawn from p on the quartic Q lie on the cubic $\Delta_1 = K = x_1\dfrac{dQ}{dx} + y_1\dfrac{dQ}{dy} + z_1\dfrac{dQ}{dz} = 0$, and the ten points where the ten tangents meet Q again lie on the quartic $(\Delta_1^2)^2 - 16\Delta\Delta_1 = U = 0$; and the form of the equation of U shows that it has double contact with K in the points where K meets the conic $\Delta_1^2 = 0$, and as K and Δ_1^2 have two coincident points at p (x_1, y_1, z_1), it follows that K and U have coincident points at p.

2499. (Proposed by S. WATSON.)—ABCD is a quadrilateral; E, F, G, H the middle points of AB, BC, CD, DA; and $m_1, m_2, \ldots m_8$ the middle points of AE, EB, BF, ... HA. Show that the sum of the squares of the 28 lines joining every two of the points $m_1, m_2, \ldots m_8$ is

$$4(AC^2 + BD^2) + AB^2 + BC^2 + CD^2 + DA^2.$$

Solution by R. TUCKER, M.A.; S. BILLS; *and others.*

Denoting all such lines as m_1m_2 by $\lambda_1, \lambda_2 \ldots \lambda_8$; as m_1m_3 by $\mu_1, \mu_2, \ldots \mu_8$; as m_1m_6 by $\nu_1, \ldots \nu_4$; we obtain, by known properties of triangles, bearing in mind that the figures EG_1, m_1m_5, m_2m_6 are parallelograms,



$$\nu_1^2 + m_1m_5^2 = 2(m_1G^2 + m_6G^2),$$

and $$\nu_2^2 + m_2m_6^2 = 2(m_2G^2 + m_6G^2);$$

therefore $$\nu_1^2+\nu_2^2+m_1m_5^2+m_2m_6^2 = \lambda_1^2+\lambda_5^2+4EG^2$$

and $$\nu_3^2+\nu_4^2+m_3m_7^2+m_4m_8^2 = \lambda_3^2+\lambda_7^2+4FH^2 \Big\} \ldots\ldots\ldots (\alpha).$$

Again, $$\mu_1^2 = \lambda_1^2+2Bm_2^2+3\lambda_2^2-2Bm_3^2$$

and $$\mu_2^2 = \lambda_3^2+2Bm_3^2+3\lambda_2^2-2Bm_2^2 \Big\};$$

therefore $$\mu_1^2+\mu_2^2 = \lambda_1^2+\lambda_3^2+6\lambda_2^2;$$

hence $$\Sigma(\mu^2) = 2(\lambda_1^2+\lambda_3^2+\lambda_5^2+\lambda_7^2)+6(\lambda_2^2+\lambda_4^2+\lambda_6^2+\lambda_8^2) \ldots\ldots (\beta).$$

Now $$(\alpha)+(\beta) = 3(\lambda_1^2+\lambda_3^2+\lambda_5^2+\lambda_7^2)+44(\lambda_2^2+\lambda_8^2);$$

and the twelve remaining squares (γ) are easily shown to be equal to

$$\lambda_1^2+\lambda_3^2+\lambda_5^2+\lambda_7^2+20(\lambda_2^2+\lambda_8^2).$$

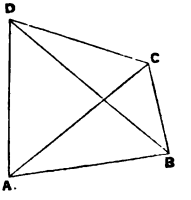
Hence the sum of the whole number of squares required is

$$(\alpha)+(\beta)+(\gamma) = 4(\lambda_1^2+\lambda_3^2+\lambda_5^2+\lambda_7^2)+64(\lambda_2^2+\lambda_8^2)$$
$$= AB^2+BC^2+CD^2+DA^2+4(AC^2+BD^2).$$

1971. (Proposed by C. W. MERRIFIELD, F.R.S.)—In a given tetrahedron, to find a set of rectangular coordinate axes, such that each axis shall pass through a pair of opposite edges.

Solution by J. J. WALKER, M.A.

Take the three edges AB, AC, AD of the tetrahedron as axes of x, y, z respectively, and let $AB=x'$, $AC=y'$, $AD=z'$. If ξ, η, ζ be the coordinates of any point (O), the plane determined by this point and the line CD will have as its equation



$$x(y'z'-\eta z'-\zeta y')+y\xi z'+z\xi y'-\xi y'z' = 0 \ldots\ldots (1),$$

since this equation is evidently satisfied by the coordinates of O, C, and D. Again, the plane determined by O and AB will have as its equation

$$y\zeta - z\eta = 0 \ldots\ldots\ldots\ldots (2).$$

Hence, if the line passing through O and intersecting the opposite edges AB, CD be $\frac{x-\xi}{l} = \frac{y-\eta}{m} = \frac{z-\zeta}{n}$, we shall have

$$l : m : n = \xi(\eta z' + \zeta y') : \eta(\eta z' + \zeta y' - y'z') : \zeta(\eta z' + \zeta y' - y'z').$$

And, by symmetry, if $\frac{x-\xi}{l'} = \frac{y-\eta}{m'} = \frac{z-\zeta}{n'}$ be the line passing through O and intersecting the opposite edges AC and BD,

$$l' : m' : n' = \xi(\xi z' + \zeta x' - x'z') : \eta(\xi z' + \zeta x') : \zeta(\xi z' + \zeta x' - x'z').$$

If now λ, μ, ν be the cosines of the angles CAD, BAD, BAC respectively, the conditions that the above two lines passing through O should be perpendicular one to the other is

$$ll' + mm' + nn' + \lambda(mn' + m'n) + \mu(l'n + ln') + \nu(lm' + l'm) = 0;$$

or, substituting from the above proportions, the condition is

$$\begin{aligned} 0 = {} & \xi^2(\xi z' + \zeta x' - x'z')(\eta z' + \zeta y') + \eta^2(\eta z' + \zeta y' - y'z')(\xi z' + \zeta x') \\ & + \zeta^2(\xi z' + \zeta x' - x'z')(\eta z' + \zeta y' - y'z') + \lambda\eta\zeta(2\xi z' + 2\zeta x' - x'z')(\eta z' + \zeta y' - y'z') \\ & + \mu\xi\zeta(2\eta z' + 2\zeta y' - y'z')(\xi z' + \zeta x' - x'z') \\ & + \nu\xi\eta\left\{(\xi z' + \zeta x')(\eta z' + \zeta y') + (\xi z' + \zeta x' - x'z')(\eta z' + \zeta y' - y'z')\right\} \ldots\ldots (3). \end{aligned}$$

If it be desired to express this condition in terms of the six edges only, we may substitute, for λ, $\frac{y'^2 + z'^2 - x''^2}{2y'z'}$, for μ, $\frac{x'^2 + z'^2 - y''^2}{2x'z'}$, for ν, $\frac{x'^2 + y'^2 - z''^2}{2x'y'}$, supposing CD $= x''$, BD $= y''$, BC $= z''$.

The condition (3) may be thrown into any one of the forms

$$\begin{aligned} 0 = {} & (\xi^2 + \eta^2 + \zeta^2 + 2\lambda\eta\zeta + 2\mu\xi\zeta + 2\nu\xi\eta)(\xi z' + \zeta x')(\eta z' + \zeta y') \\ & - (\xi^2 + \zeta^2 + \lambda\eta\zeta + 2\mu\xi\zeta + \nu\xi\eta)(\eta z' + \zeta y')x'z' \\ & - (\eta^2 + \zeta^2 + 2\lambda\eta\zeta + \mu\xi\zeta + \nu\xi\eta)(\xi z' + \zeta x')y'z' \\ & + (\zeta^2 + \lambda\eta\zeta + \mu\xi\zeta + \nu\xi\eta)x'y'z'^2 \ldots\ldots\ldots\ldots\ldots\ldots (4), \end{aligned}$$

$$\begin{aligned} 0 = {} & \xi(\xi + \nu\eta + \mu\zeta)(\xi z' + \zeta x' - x'z')(\eta z' + \zeta y') \\ & + \eta(\nu\xi + \eta + \lambda\zeta)(\eta z' + \zeta y' - y'z')(\xi z' + \zeta x') \\ & + \zeta(\mu\xi + \lambda\eta + \zeta)(\xi z' + \zeta x' - x'z')(\eta z' + \zeta y' - y'z') \\ & + \nu\left\{(\xi z' + \zeta x')y' + (\eta z' + \zeta y')x' + x'y'z'\right\}\xi\eta z' \ldots\ldots\ldots (5), \end{aligned}$$

$$\begin{aligned} 0 = {} & x'y'\zeta^4 + \left\{(\xi y' + \eta x')z' + 2(\mu\xi + \lambda\eta - z')x'y'\right\}\zeta^3 \\ & + \left\{(\xi - x')(\eta - y')z'^2 + (\xi^2 + \eta^2)x'y' + (\mu\xi + \lambda\eta)(2\xi y' + 2\eta x' - 3x'y')z'\right\}\zeta^2 \\ & + \left\{(\xi^2 + \eta^2 + 2\nu\xi\eta)(\xi y' + \eta x' - x'y')z' + [(\mu\xi + \lambda\eta)(2\xi\eta + x'y' - \xi y' - \eta x') \right. \\ & \qquad \left. - (\mu x' + \lambda y')\xi\eta]z'^2\right\}\zeta \\ & + \left\{\xi(\xi - x') + \eta(\eta - y') + \nu[(\xi - x')(\eta - y') + \xi\eta]\right\}\xi\eta z'^2 \ldots\ldots\ldots (6). \end{aligned}$$

It appears from the above investigation that the locus of a point determined by the condition that two lines meeting in it shall be perpendicular one to the other (or, generally, be inclined at any given angle), these lines resting, one on two fixed lines in space, the other on two other fixed lines intersecting each one of the former pair, is a quartic surface. Hence the origin of the three rectangular axes referred to in the question may be any one of the points common to three quartic surfaces; viz., that to which (4) is the equation, and two others whose equations are found from (4) by interchanging ξ and ζ, η and ζ successively.

Four points common to these surfaces are, plainly, the four corners of the tetrahedron. The directions of the three axes drawn through the point A will be found by determining ll'', mm', $n'n''$ from the six equations

$$\left.\begin{aligned} mm' + \lambda mn' + \mu ln' + \nu lm' &= 0 \\ ll'' + \lambda mn'' + \mu ln'' + \nu l''m &= 0 \\ n'n'' + \lambda m'n'' + \mu l''n' + \nu l''m' &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} l^2 + m^2 + 2\nu lm - 1 &= 0 \\ m'^2 + n'^2 + 2\lambda m'n' - 1 &= 0 \\ l''^2 + n''^2 + 2\mu l''n'' - 1 &= 0 \end{aligned}\right\},$$

the equations to the axes being $\frac{x}{l} = \frac{y}{m}$, $\frac{y}{m'} = \frac{z}{n'}$, $\frac{z}{n''} = \frac{x}{l''}$.

In this case, the construction of the three axes may be conceived geometrically thus:—At A raise a perpendicular P to the plane ABC, meeting a sphere with centre A and radius AD in the point Q; through P two planes may be drawn, each cutting the planes ADC, ABD in a pair of rectangular right lines, (for the solution of this problem would be effected by drawing through Q a great circle of the sphere such that its arc intercepted between the circles in which the planes ADC, ABD cut the sphere may be a quadrant). A right line in the plane ABC, perpendicular to either of these planes through P, would form the third rectangular axis to the corresponding pair in the planes ACD, ABD.

[A Solution by *Quaternions* is given on p. 85 of Vol. VIII. of the *Reprint*.]

2696. (Proposed by the Rev. T. P. KIRKMAN, M.A., F.R.S.) — Let $G = 1 + \theta + \theta^2 + \theta^3$ be any group of powers of a substitution made with the elements 1234. Let H_1 be any asymmetric function of $x_1x_2x_3x_4$. And let $\theta\,;H_1$ or $G\,;H_1$ be the symbol of the result of operation on H_1 by θ or by the whole group G. Thus $G\,;H_1$ is an algebraic sum of four values of H_1.

1. There is a substitution ϕ such that $\left\{(G\,;H_1^i)^j + \phi\,;(G\,;H_1^i)^j\right\}^k$ is a function which has not more than three values, all given with G and H_1, whatever be the integer exponents i, j, k.

2. Hereby the algebraic solution of the general quintic equation is attainable at the cost of few words.

Solution by the PROPOSER.

Let the roots of the quintic $\Theta = 0$ be $x_0x_1x_2x_3x_4$. We can write, if α be an imaginary fifth root of unity,

$$x_0 = \Sigma x + (\alpha + \alpha^2 + \alpha^3 + \alpha^4)(x_1 + x_2 + x_3 + x_4), \text{ or}$$

$$x_0 = \tfrac{1}{5}\Sigma x + \tfrac{1}{5}(x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 + \alpha^4 x_4)^{\frac{5}{5}} + \tfrac{1}{5}(x_0 + \alpha x_2 + \alpha^2 x_3 + \alpha^3 x_4 + \alpha^4 x_1)^{\frac{5}{5}},$$
$$+ \tfrac{1}{5}(x_0 + \alpha x_3 + \alpha^2 x_4 + \alpha^3 x_1 + \alpha^4 x_2)^{\frac{5}{5}} + \tfrac{1}{5}(x_0 + \alpha x_4 + \alpha^2 x_1 + \alpha^3 x_2 + \alpha^4 x_3)^{\frac{5}{5}},$$

which, by ordinary involution, becomes

$$x_0 = \tfrac{1}{5}\Sigma x + \tfrac{1}{5}(S + H_1)^{\frac{1}{5}} + \tfrac{1}{5}(S + H_2)^{\frac{1}{5}} + \tfrac{1}{5}(S + H_3)^{\frac{1}{5}} + \tfrac{1}{5}(S + H_4)^{\frac{1}{5}},$$

where S is symmetric in $x_0x_1x_2x_3x_4$, and H_1 is a 24-valued function of the form

$$H_1 = u_0 + \alpha u_1 + \alpha^2 u_2 + \alpha^3 u_3 + \alpha^4 u_4,$$

which is invariable by the cyclical permutation of $x_0x_1x_2x_3x_4$, and consequently receives all its 24 values by the permutations of any four of them, for example by those of $x_1x_2x_3x_4$. H_2, H_3, and H_4 are made from H_1 by the cyclical permutations of $x_1x_2x_3x_4$, so that, if G be the group 1234, 2341, 3412, 4123, we have $G;H_1 = H_1 + H_2 + H_3 + H_4$,

the root x_0 being undisturbed by the substitutions of G_1; that is, $G;H$ is a function of x_1, x_2, x_3, x_4, invariable by the substitutions of G, and equally so is

$$G;H_1^i = H_1^i + H_2^i + H_3^i + H_4^i = K_i,$$

whatever be the integer i; for the substitution which changes H_π into H_m will change $H_\pi . H_\pi$ into $H_m . H_m$ and H_π^i into H_m^i. And it is evident that

$$(G;H_1^i)^j = (H_1^i + H_2^i + H_3^i + H_4^i)^j = K_i^j$$

is invariable by the substitutions of G; for all its j factors are so: in other words, K_i^j is a function formed upon the group G, of the fourth order. The group of the eighth order $P = G + \phi;G = \begin{matrix}1234 + 1432 = G + {}_1G \text{ is com-}\\ 2341 \quad 4321\\ 3412 \quad 3214\\ 4123 \quad 2143\end{matrix}$

posed of G and its derivate ${}_1G$ by any one ϕ of the four substitutions in ${}_1G$;

and $$K_i^j + \phi;K_i^j = K_i^j + {}_1K_i^j = L_j$$
$$= (H_1^i + H_2^i + H_3^i + H_4^i)^j + (H_5^i + H_6^i + H_7^i + H_8^i)^j,$$

in which eight of the 24 values of H_1 appear, is a function formed on the group P and invariable by its eight substitutions.

This follows from a property of every group $R = G + {}_1G + {}_2G + \ldots + {}_mG$ composed of a group G of any order and of m derivates of G, that if θ be any substitution of the group R, the operation $\theta;R$ can change nothing in the right member except the order of $G, {}_1G \ldots {}_mG$; whence it follows that, if $(G), ({}_1G), \ldots ({}_mG)$ be functions formed on G and its m derivates, the function

$$(R)_i = (G)^i + ({}_1G)^i + \ldots + ({}_mG)^i$$

is invariable by the substitutions of R, and any power of (R_i) is the like; that is, $$(K_i^j + {}_1K_i^j)^k = L_j^k$$

is a function invariable by the 8 substitutions of P, whatever be the integers k, j, i.

It follows that L_j^k is a three-valued function; for if $\lambda = 2314$ and $\lambda^2 = 3124$,

$$\Omega = P + \lambda;P + \lambda^2;P = P + {}_1P + {}_2P$$

is the whole group of 1.2.3.4 substitutions made with 1234. The sum of the three values of L_j^k made on P and its two derivates ${}_1P$ and ${}_2P$, viz.

$$\Sigma_k = L_j^k + {}_1L_j^k + {}_2L_j^k = L_j^k + \lambda;L_j^k + \lambda^2;L_j^k,$$

is therefore invariable by the 24 substitutions of Ω; that is, Σ_k is a function of $x_0x_1x_2x_3x_4$, which is symmetrical in the four roots $x_1x_2x_3x_4$: it is also a function of the 24 values of H_1, every one of which is invariable by the

cyclical permutations of $x_0x_1x_2x_3x_4$; but this does not in general make Σ_k invariable by them. Σ_k is in fact a five-valued function of $x_0x_1x_2x_3x_4$.

The function Σ_k may be thus symbolized,

$$\Sigma_k = G_3\,;\, G_2^k\,;\, G_4^j\,;\, H_1^i,$$

where G_4; $H_1^i = K_i^j$ is the jth power of the sum of four values of H_1^i formed on G above written of the 4th order; where $G_2^k\,;G_4^j\,;H_1^i = G_2^k\,;K_i^j = L_j^k$ is the kth power of the sum of two values of K_i^j made on $G_2 = 1234 + 1432$ of the second order, and where Σ_k is the sum of the three values of L_j^k formed on the group $G_3 = \begin{matrix}1234\\2314\\3124\end{matrix}$ of the third order. Thus Σ_k is the sum of the kth powers of three values of L_j, the sum of the jth powers of two values of K_i, the sum of the ith powers of four values of our given function H_1.

I find that it is more convenient for higher applications to form the function

$$\Sigma_\beta = \Gamma_4\,;\,\Gamma_3^\beta\,;\,\Gamma_2^\alpha\,;\,H_1^i = \Gamma_4\,;\,Z_\alpha^\beta,$$

where Γ_4 is the above written G, Γ_3 is the above G_3, and Γ_2 is $1234 + 2143$; *i. e.*, Γ_r is the group of cyclical permutations of $1, 2, \ldots r$. This Σ_β is the sum of the βth powers of four values of $Z_\alpha = \Gamma_3\,;\,Y_i^\alpha$, which is the sum of αth powers of three values of $Y_i = \Gamma_2\,;\,H_1^i$, which is the sum of ith powers of two values of our H_1. It follows directly from the above stated property of any group R, that Σ_β is a symmetrical function of $x_1x_2x_3x_4$, and also that

$$\Sigma_\mu = \Gamma_n\,;\,\Gamma_{n-1}^\alpha\,;\,\Gamma_{n-2}^\lambda \ldots\ \Gamma_4^\gamma\,;\,\Gamma_3^\beta\,;\,\Gamma_2^\alpha\,;\,H_1^i$$

is a symmetrical function of $x_1x_2x_3 \ldots x_n$, if H_1 be any asymmetric function of these variables, whatever be the positive exponents $\mu, \lambda, \ldots \gamma, \beta, \alpha, i$.

Let Σ_β be formed for the 24 combinations of $i < 3$, $\alpha < 4$, $\beta < 5$. We can write, for any fixed pair of values of α and i (there are six pairs),

$$\begin{array}{l|l} \Sigma_1 = Z_\alpha + {}_1Z_\alpha + {}_2Z_\alpha + {}_3Z_\alpha & \Sigma_3 = Z_\alpha^3 + {}_1Z_\alpha^3 + {}_2Z_\alpha^3 + {}_3Z_\alpha^3 \\ \Sigma_2 = Z_\alpha^2 + {}_1Z_\alpha^2 + {}_2Z_\alpha^2 + {}_3Z_\alpha^2 & \Sigma_4 = Z_\alpha^4 + {}_1Z_\alpha^4 + {}_2Z_\alpha^4 + {}_3Z_\alpha^4 \end{array}$$

of which the first line shows four values of Z_α, the second four values of Z_α^2, &c., made on Γ_4.

The first members are all rational functions of $x_0x_1x_2x_3x_4$, and symmetrical in $x_1x_2x_3x_4$, the separate terms on the right are asymmetric functions of $x_0x_1x_2x_3x_4$. These four conditions enable us to transform Z_α into the *issuing root* $\left\{Z_\alpha\right\}$ of a quartic whose coefficients are given rational and symmetrical functions of $x_1x_2x_3x_4$, and the asymmetric Z_α is thus changed into an irrational function of $x_0x_1x_2x_3x_4$, which is symmetric in $x_1x_2x_3x_4$.

This $\left\{Z_\alpha\right\}$ we can thus obtain both for $i=1$ and for $i=2$; and, for a fixed value of i, we can write

$$\left\{Z_1\right\} = Y_i + {}_1Y_i + {}_2Y_i, \quad \left\{Z_2\right\} = Y_i^2 + {}_1Y_i^2 + {}_2Y_i^2, \quad \left\{Z_3\right\} = Y_i^3 + {}_1Y_i^3 + {}_2Y_i^3,$$

where we read on the right the values of Y_i^a formed on Γ_3. Hence Y_i can be written as the issuing root $\{Y_i\}$ of a given cubic, whose coefficients are given irrational functions of $x_0x_1x_2x_3x_4$, symmetrical in $x_1x_2x_3x_4$; and if we take care that the issuing roots $\{Z_1\}$, $\{Z_2\}$, $\{Z_3\}$ are corresponding roots of the quartics which yield them, the coefficients of this cubic will be unambiguous. We have two such cubics to solve, whose corresponding roots are

$$\{Y_1\} = H_1 + {}_1H_1, \quad \{Y_2\} = H_1^2 + {}_1H_1^2,$$

and H_1 is finally transformed into $\{H_1\}$, the root of a given quadratic, whose coefficients are irrational and unambiguous functions of $x_0x_1x_2x_3x_4$, symmetrical in $x_1x_2x_3x_4$. Let us write this $H_1 = \{{}_0H_1\}$, showing that H_1 is transformed into an irrational symmetric in all the roots except x_0.

The function H_1 of 24 values receives them all by the permutations of any four roots. In the preceding process, we have left x_0 undisturbed, treating it as a constant in our substitutions. We can repeat the process, and permuting only $x_0x_2x_3x_4$, leave x_1 undisturbed, forming our groups with 2340 instead of 1234. The result will be $H_1 = \{{}_1H_1\}$, or the same asymmetric H_1 is transformed into an irrational function of $x_0x_1x_2x_3x_4$, symmetrical in all except x_1. And we can thus obtain three more results

$$H_1 = \{{}_2H_1\}, \quad H_1 = \{{}_3H_1\}, \quad H_1 = \{{}_4H_1\},$$

where $\{{}_rH_1\}$ is symmetrical in all the roots except x_r. From these we deduce $\quad H_1^5 = \{{}_0H_1\}\cdot\{{}_1H_1\}\cdot\{{}_2H_1\}\cdot\{{}_3H_1\}\cdot\{{}_4H_1\} = [H_1]$,

which is symmetrical in all the five roots $x_0x_1x_2x_3x_4$, and $H_1 = [H_1]^{\frac{1}{5}}$ is the required transformation of H_1 into a symmetrical function of the roots of $\Theta=0$. With equal ease we can obtain, by our five-fold process, the transformations $\quad H_2 = [H_2]^{\frac{1}{5}}, \quad H_3 = [H_3]^{\frac{1}{5}}, \quad H_4 = [H_4]^{\frac{1}{5}}$;

and it is highly probable that we shall obtain four values at once in each process, merely by varying the reading of the radicals in that root $\{Z_\alpha\}$ of the quartic that we chose out of four.

Wherefore nothing but a most monstrous amount of algebraic labour can prevent our writing the root of $\Theta=0$ thus, in terms of $x_0x_1x_2x_3x_4$,

$$x_0 = \tfrac{1}{5}\Sigma x + \tfrac{1}{5}(S + [H_1]^{\frac{1}{5}})^{\frac{1}{5}} + \tfrac{1}{5}(S + [H_2]^{\frac{1}{5}})^{\frac{1}{5}} + \tfrac{1}{5}(S + [H_3]^{\frac{1}{5}})^{\frac{1}{5}} + \tfrac{1}{5}(S + [H_4]^{\frac{1}{5}})^{\frac{1}{5}},$$

which is an irrational and a symmetrical function of all the roots.

Is this now an algebraical solution of the quintic $\Theta=0$? It is, or it is not, according as the irrationals $[H_1]$. $[H_2]$, $[H_3]$, $[H_4]$ are, or are not, for the reason that they are symmetrical in the five roots, reducible in a finite form into functions of the coefficients of $\Theta=0$.

If this reduction can be effected, as I believe it can, whatever be the number of the roots $x_0, x_1, \ldots x_n$, then assuredly the great problem of equations is solved for all degrees.

The five groups of 24 which we employ in the five processes for obtaining the quadratic roots, $\{{}_0H_1\}$, $\{{}_1H_1\}$, $\{{}_2H_1\}$, $\{{}_3H_1\}$, $\{{}_4H_1\}$, transforma-

tions of the same H_1, are got by cyclically permuting $x_0x_1x_2x_3x_4$ in any of the five groups; and they complete together the entire group of 120, each having one undisturbed element. Hence there is little doubt that from $\{_0H_1\}$ the remaining four irrationals will be got by cyclically permuting the five roots, without repetition of our process; while the five irrationals $\{_0H_2\}$, $\{_1H_2\}$, $\{_2H_2\}$, $\{_3H_2\}$, $\{_4H_2\}$, as well as two more fives, will be obtained by varying the reading of our quartic root $\{Z_a\}$. The five tactical values of $\{_0H_1\}$, being all numerically the same, lead to the solution of the binomial quintic $H_1^5 = [H_1]$, and the prognostic of the acute MURPHY and others is fulfilled for equations of all degrees.

The preceding investigation, even if a doubt remains, will, I hope, justify my proposal of Question 2696.

2511. (Proposed by Professor SYLVESTER.)—If on an ordinary chessboard we compare any two rows or any two columns, squares opposite one another will be all of unlike or all of like colours according as the number of rows or columns intervening is even (zero included) or odd.

1. Show that it is possible to paint the squares in a manner such that, on comparing any two rows or columns, four of the squares opposite one another shall have like and the remaining four unlike colours.

2. Extend the construction to the case of a chessboard containing 2^i (instead of eight) squares in each direction.

Solution by G. A. OGILVIE.

The principles upon which I have made this depend are as follows:—

(1.) There are only two ways of painting a chessboard containing four squares, so that, on comparing either of the two pairs of columns, there shall be one like and one unlike colour; viz., either with three white and one black or with three black and one white.

(2.) This property can be extended to the case of a board containing any number of squares when the board is made up of three similar groups of squares and one dissimilar group.

[By "group" is meant the fourth part of a square, made by joining middle points of sides; and by "dissimilar," that all the colours are changed so that the blacks become whites and the whites blacks.]

(3.) If in a square, as mentioned in (1), containing two in each side, we consider that the first vertical column a and the second vertical column b should become $-a$ and $-b$ when the colours in each column are changed; then we can form a square of four in each side of the required species, containing a and $-b$ in the 1st column, b and $-a$ in the 2nd, $-a$ and $-b$ in the 3rd, b and a in the 4th; and this is evidently symmetrical with regard

to its horizontal columns. Let B stand for a black and W for a white square;

then $\begin{matrix}B\\B\end{matrix}$ is the column represented by a, or $\begin{matrix}W\\W\end{matrix}$, or &c.

and $\begin{matrix}W\\W\end{matrix}$ is the column represented by $-a$, or $\begin{matrix}B\\B\end{matrix}$, or &c.

The two squares of four in each side, obtained by the processes indicated in (2) and (3) are (Fig. 1) $\begin{matrix}B&W&B&W\\B&B&B&B\\B&W&W&B\\B&B&W&W\end{matrix}$ (Fig. 2) $\begin{matrix}B&W&W&W\\B&B&W&B\\B&W&B&B\\W&W&W&B\end{matrix}$

where the original square is $\begin{matrix}B&W\\B&B\end{matrix}$.

I shall now proceed to show (i) how a square of 8 in each side can be formed subject to the condition that, on comparing any two rows, &c.; firstly, with the number of black and white squares the same; secondly, when it is not the same; and (ii) to show that this can be extended by induction to the case of a chessboard containing 2^i in each side.

(i)$_\alpha$. To prove it is true for a square of 8 when the number of black and of white is to be the same. Let us take three squares (Fig. 1) of four in each side to represent three whites; and a square with all the colours unlike those in Fig. 1 to represent a black, thus $\begin{matrix}A_1&B_1\\A_2&A_3\end{matrix}$ where B_1 is put for $\begin{matrix}W&B&W&B\\W&W&W&W\\W&B&B&W\\W&W&B&B\end{matrix}$

Let these 3 whites be represented by A_1, A_2, A_3, and the black by B_1, as seen in the figure. Then, on comparing any row in $A_1 + A_2$, say the 3rd, with any row in $B_1 + A_3$, there will be 4 like and 4 unlike.

Firstly, let us take the 3rd row in $A_1 + A_2$ with the 3rd row in $B_1 + A_3$:

the 3rd in A_1 has all (*i. e.* 4) unlike the 3rd in B_1, by hypothesis;
the 3rd in A_2 has all like the 3rd in A_3;

therefore the 3rd in $A_1 + A_2$ has 4 like and 4 unlike the 3rd in $B_1 + A_3$.

Secondly, let us take any row in $B_1 + A_3$:

the 3rd in A_1 has 4 unlike the 3rd in B_1 which has 2 unlike the nth in B_1;
the 3rd in A_2 has 4 like the 3rd in A_3 which has 2 like the nth in A_3;

therefore the 3rd in $A_1 + A_2$ has 4 like the nth in $B_1 + A_3$,
also the 3rd in $A_1 + A_2$ has 4 like the nth in $A_1 + A_2$, by hypothesis.

Similarly the mth in $A_1 + A_2$ has 4 like the nth in $B_1 + A_3$.
and the mth in $A_1 + A_2$ has 4 like the mth in $A_1 + A_2$;

therefore the mth row has 4 like and 4 unlike in any column.

The proof is identical for horizontal columns. Therefore, on comparing any two columns, &c.

(i)$_\beta$ To show how to make the square of 8 in each side when the number of black and white is to be the same. The construction and proof of this are the same as before, merely using Fig. 2 for Fig. 1. We shall then get a

figure of the kind marked Fig. 3 below, whereas in the other case we get a figure similar to that marked Fig. 4.

B	W	W	W	B	W	W	W
B	B	W	B	B	B	W	B
B	W	B	B	B	W	B	B
W	W	W	B	W	W	W	B
B	W	W	W	W	B	B	B
B	B	W	B	W	W	B	W
B	W	B	B	W	B	W	W
W	W	W	B	B	B	B	W

FIG. 3.

B	W	B	W	B	W	B	W
B	B	B	B	B	B	B	B
B	W	W	B	B	W	W	B
B	B	W	W	B	B	W	W
B	W	B	W	W	B	W	B
B	B	B	B	W	W	W	W
B	W	W	B	W	B	B	W
B	B	W	W	W	W	B	B

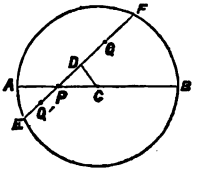
FIG. 4.

(ii.) To show how this can be extended to the case of a chessboard containing 2^i squares in each direction. It can be shown in a similar manner to the proof of the square of 8 in each side, that if it holds for a square containing 2^n, it holds for 2^{n+1} in each side. But it holds for 4 and 8, therefore it holds universally for a board containing 2^i in each side.

2407. (Proposed by the EDITOR.)—Show (1) that the average distance of all the pairs of points that can be taken within a circle, in parts of the radius, is $\frac{128}{45\pi}$; and (2) that the same average for a sphere is $\frac{36}{35}$.

Solution by STEPHEN WATSON.

1. Let C be the centre of the given circle, P and Q any two points within it, and AB a diameter through P. Put $CA = a$, $CP = x$, $PQ = y$, and $\angle QPB = \phi$. Then an element of the circle at P $= 2\pi x\,dx$, and at Q $= y\,d\phi\,dy$; and the number of positions of P and Q is $\pi^2 a^4$. The limits of x are 0 and a, of ϕ are 0 and $\frac{1}{2}\pi$ and the result doubled, and of y are 0 and $(a^2 - x^2\sin^2\phi)^{\frac{1}{2}} + x\cos\phi$, when Q lies on PF; but when Q lies on PE, as at Q′, and $PQ' = y$, the limits are 0 and $\left\{(a^2 - x^2\sin^2\phi)^{\frac{1}{2}} - x\cos\phi\right\}$; thus the required average is



$$\frac{4}{\pi a^4}\int_0^{\frac{1}{2}\pi} d\phi \int_0^a x\,dx \int y^2\,dy,$$

which, when integrated for y in the manner stated above, becomes

$$= \frac{4}{\pi a^4}\int_0^{\frac{1}{2}\pi} d\phi \int_0^a \left\{\tfrac{2}{3}(a^2 - x^2\sin^2\phi)^{\frac{3}{2}} + 2x^2\cos^2\phi\,(a^2 - x^2\sin^2\phi)^{\frac{1}{2}}\right\} x\,dx$$

$$= \frac{8a}{15\pi}\int_0^{\frac{1}{2}\pi} \frac{1 + \cos^2\phi - 6\cos^5\phi + 4\cos^7\phi}{\sin^4\phi}\,d\phi = \frac{128a}{45\pi}.$$

2. In this case an element at $P = 4\pi x^2 dx$, and at $Q = 2\pi y \sin\phi\, y\, d\phi\, dy$; and the total positions of P and Q is $\frac{1}{2}\pi^2 a^6$. The limits are as before, except that the integral for ϕ must not be doubled; hence the average is

$$\frac{9}{2a^6}\int_0^{\frac{1}{2}\pi} \sin\phi\, d\phi \int_0^a x^2\, dx \int y^2\, dy,$$

which, integrated for y as in the preceding case, becomes

$$= \frac{9}{4a^6}\int_0^{\frac{1}{2}\pi} \sin\phi\, d\phi \int_0^a \left\{(a^2 - x^2\sin^2\phi)^2 + 6x^2\cos^2\phi\,(a^2 - x^2\sin^2\phi) + x^4\cos^4\phi\right\} \times x^2\, dx$$

$$= \frac{6a}{35}\int_0^{\frac{1}{2}\pi} (1 + 6\cos^2\phi + 15\cos^4\phi)\sin\phi\, d\phi = \frac{36a}{35}.$$

2702. (Proposed by W. S. McCay, B.A.)—The locus, in areal coordinates, of the centre of an ellipse, of constant area (πn^2), circumscribing a triangle, is $(x+y+z)(x+y-z)(x-y+z)(-x+y+z) = \dfrac{4x^2y^2z^2}{n^4}$.

I. *Solution by the* Rev. J. L. Kitchin, M.A.

If the conic be given in areal coordinates by the equation $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$, the centre (x, y, z) will be given by the equations

$$\frac{x}{l(l-m-n)} = \frac{y}{m(m-n-l)} = \frac{z}{n(n-l-m)} = \frac{2\Delta}{l^2+m^2+n^2-2mn-2nl-2lm} \quad \ldots(1).$$

Now the area (Whitworth's *Modern Geometry*, p. 281) $= \dfrac{2\pi\Delta H}{(-K)^{\frac{3}{2}}} = \pi n^2$,

where $K = \frac{1}{4}(l^2+m^2+n^2-2mn-2nl-2lm)$, and $H = \frac{1}{4}lmn$.

From the values of x, y, z, given by (1), we readily obtain

$$(x+y+z)(x+y-z)(x-y+z)(-x+y+z) = \frac{\Delta^4(l-m-n)^2(m-n-l)^2(n-l-m)^2}{-4K^3};$$

also $$\left(\frac{\Delta}{2K}\right)^6 = \frac{x^2y^2z^2}{l^2m^2n^2(l-m-n)^2(m-n-l)^2(n-l-m)^2};$$

$$\therefore (x+y+z)(x+y-z)(x-y+z)(-x+y+z)\frac{\Delta^6}{(2K)^6} = \frac{-\Delta^4x^2y^2z^2}{64H^2K^3};$$

$$\therefore (x+y+z)(x+y-z)(x-y+z)(-x+y+z) = x^2y^2z^2\left(\frac{-K^3}{\Delta^2H^2}\right) = \frac{4x^2y^2z^2}{n^4}.$$

II. *Solution by the* Proposer.

Using the notation of the eccentric angle, the central perpendicular on

$\frac{x}{a}\cos\frac{\beta+\gamma}{2}+\frac{y}{b}\sin\frac{\beta+\gamma}{2}=\cos\frac{\beta-\gamma}{2}$ is $X=\frac{ab}{b'}\cos\frac{\beta-\gamma}{2}$, where b' is the parallel semi-diameter. But calling d, e, f the sides of the given triangle, we know (Salmon's *Conics*, p. 205) that $d=2b'\sin\frac{1}{2}(\beta-\gamma)$, therefore

$$\frac{dX}{ab}=\sin(\beta-\gamma);\quad \text{similarly } \frac{eY}{ab}=\sin(\gamma-\alpha),\quad \text{and } \frac{fZ}{ab}=\sin(\alpha-\beta).$$

But if $\theta+\theta'+\theta''=0$, it is easy to show that

$$(\sin\theta+\sin\theta'+\sin\theta'')(\sin\theta+\sin\theta'-\sin\theta'')(\sin\theta-\sin\theta'+\sin\theta'')\times(-\sin\theta+\sin\theta'+\sin\theta'')=4\sin^2\theta\sin^2\theta'\sin^2\theta''.$$

Hence we obtain at once the above equation for the locus.

2631. (Proposed by W. H. LAVERTY, B.A.)—A conic F has double contact with one of a set of conics having double contact with one another. Show that the chords of intersection of F with each of the set will intersect on the common chord of contact of the whole set.

Solution by the PROPOSER.

Project the set into a series of concentric circles, and the conic into a conic having double contact with one of the circles. Therefore, the common centre of the circles will lie on one of the principal axes of the conic; consequently, all the chords of intersection will be parallel, *i. e.*, will intersect on the line at infinity which is the common chord of contact of the circles.

2267. (Proposed by W. H. LAVERTY.)—If ∇ represent the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and if a', b', c', f', g', h', be the minors which respectively are the coefficients of a, b, c, f, g, h, in the expansion of ∇: then show that the coordinates (x, y) of the foci of $(abcfgh)(xy1)^2=0$, are given by

$$2c'x-2g'=\nabla^{\frac{1}{2}}\left\{(a-b+2hi)^{\frac{1}{2}}+(a-b-2hi)^{\frac{1}{2}}\right\},$$

and

$$2c'yi-2f'i=\nabla^{\frac{1}{2}}\left\{(a-b+2hi)^{\frac{1}{2}}-(a-b-2hi)^{\frac{1}{2}}\right\},$$

where i represents $\sqrt{-1}$. Also show that $(c'x-g')(c'y-f')=\Delta h$ is the equation to an hyperbola concentric with the given conic and passing through its four foci.

Solution by the PROPOSER.

The foci are the intersections of the pairs of tangents drawn from the circular points at infinity to the conic. Now the coordinates of these circular points are

$$\frac{x}{i}=\frac{y}{1}=p,\quad \text{and}\quad \frac{x}{-i}=\frac{y}{1}=q;$$

where p and q are infinite; and therefore in any equation involving no other infinite quantities, we must neglect all but the higher powers of p or q, and then divide through by that power. Remembering this, we get for the pair of tangents from the first circular point

$$(ax^2+by^2+2hxy+2gx+2fy+c)\ (-a+2hi+b) = (iu_x+u_y)^2,$$

whence, reducing and solving as a quadratic in $(x-yi)$, we find

$$(ab-h^2)\ (x-yi)-(g'-if') = \pm\nabla^{\frac{1}{2}}(a-b-2hi)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots(1).$$

Similarly for the other two we get

$$(ab-h^2)\ (x+yi)-(g'+if') = \pm\nabla^{\frac{1}{2}}(a-b+2hi)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots(2).$$

Adding and subtracting these alternately, we get the two required equations.

Also $$(2)^2-(1)^2 = 4i\,(c'x-g')\ (c'y-f') = 4ih\nabla,$$

another form of which will be $c'xy-g'y-f'x+h' = 0$.

The expressions for the coordinates of the foci simplify very much when $h = 0$.

2624. (Proposed by M. JENKINS, B.A.)—If ABC, A″B″C″ be two triangles, and T any point such that the area of the triangle $(\alpha\beta\gamma)$ formed by the intersections (α, β, γ) of TA″, TB″, TC″ respectively with BC, CA, AB, is constant; then the locus of T is a cubic having only one arbitrary parameter.

I. *Solution by the* PROPOSER.

For if h be the given constant and (C_h) denote the required locus, (C_h) passes through the intersections of (C_0) and (C_∞). Now if α', β', γ' be the intersections of BC, B″C″, &c., (C_0) is evidently a locus passing through A, B, C, A″, B″, C″, α', β', γ', and it may be readily shown that the locus cannot cut the straight line BC in more points than B, C, and α'; therefore C_0 is a cubic; and it will *in general* be defined by the nine points named. Also (C_∞) is the three straight lines through A″, B″, C″, drawn parallel to BC, CA, AB, respectively, whence follows the proposition given above.

When A″, B″, C″ are in the same straight line, α', β', γ' are also in the straight line A″B″C″, and the nine points named above are not sufficient to determine the cubic (C_0). But if D, E, F be the intersections of BC″, CB″, &c., then, in the general case, D, E, F are also points on (C_0). Hence, in the particular case, (C_0) breaks up into the straight line A″B″C″ and the conic S_0 (ABCDEF). These six points must lie on a conic, because (AF, CD), (BD, AE), (CE, BF) are three collinear points. If we now fix the positions of the points A″, B″, C″ on the straight line A″B″C″, so that they shall be in involution with α', β', γ', we shall find that EF passes through α', &c., and the conic through P, Q, the foci of the involution; and if A′, B′, C′ be the poles of BC, CA, AB, respectively, that A″A′, B″B′, C″C′ pass through (R) the pole of A″B″C″PQ, and thus obtain a generalization of Quest. 2395.

(C_h) does not necessarily break up because (C_0) does; but if A″B″C″ be the line at infinity, it gives an indeterminate area, and forms part of the

(C_0), (C_∞), and (C_h) loci. Hence, in this case, the (C_h) cubic breaks up into the line at infinity and a conic. The conic passes through the six points of intersection with the original conic of three straight lines, which are parallel to BC, CA, AB, respectively, and are at an infinite distance; that is, the conic is similar to and coaxial with the original conic.

II. *Solution by* J. J. WALKER, M.A.

Let ABC be taken as the triangle of reference, and the trilinear coordinates of T, A″, B″, C″ be (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ respectively; then the coordinates of the intersections of TA″ with BC, of TB″ with AC, and of TC″ with AB, will be

$$\begin{array}{ccc} 0, & \dfrac{S(\alpha\beta_1-\beta\alpha_1)}{S_1\alpha-S\alpha_1}, & \dfrac{S(\alpha\gamma_1-\gamma\alpha_1)}{S_1\alpha-S\alpha_1}, \\ \dfrac{S(\beta\alpha_2-\alpha\beta_2)}{S_2\beta-S\beta_2}, & 0, & \dfrac{S(\beta\gamma_2-\gamma\beta_2)}{S_2\beta-S\beta_2}, \\ \dfrac{S(\gamma\alpha_3-\alpha\gamma_3)}{S_3\gamma-S\gamma_3}, & \dfrac{S(\gamma\beta_3-\beta\gamma_3)}{S_3\gamma-S\gamma_3}, & 0, \end{array}$$

where $2S = a\alpha+b\beta+c\gamma$, $2S_1 = a\alpha_1+b\beta_1+c\gamma_1$, $2S_2 = a\alpha_2+\ldots$ $2S_3 = a\alpha_3+\ldots$

The area of the triangle formed by lines joining these three points of intersection is

$$\frac{abc\,S}{8}\,\frac{(\gamma\beta_3-\beta\gamma_3)(\alpha\gamma_1-\gamma\alpha_1)(\beta\alpha_2-\alpha\beta_2)+(\beta\gamma_2-\gamma\beta_2)(\gamma\alpha_3-\alpha\gamma_3)(\alpha\beta_1-\beta\alpha_1)}{(S_1\alpha-S\alpha_1)(S_2\beta-S\beta_2)(S_3\gamma-S\gamma_3)},$$

and the numerator of the latter factor I find can be thrown into the form $[(\beta_3\gamma_2)\ldots$ meaning $(\beta_3\gamma_2-\beta_2\gamma_3)\ldots]$

$$\Sigma = \alpha_1\{(\beta_3\gamma_2)\alpha+(\gamma_3\alpha_2)\beta+(\alpha_3\beta_2)\gamma\}\beta\gamma+\beta_2\{(\gamma_3\beta_1)\alpha+(\alpha_3\gamma_1)\beta+(\beta_3\alpha_1)\gamma\}\gamma\alpha$$
$$+\gamma_3\{(\beta_2\gamma_1)\alpha+(\gamma_2\alpha_1)\beta+(\alpha_2\beta_1)\gamma\}\alpha\beta+\{\alpha_1(\beta_2\gamma_3)+\beta_1(\gamma_3\alpha_2)+\gamma_1(\alpha_2\beta_3)\}\alpha\beta\gamma.$$

If the area of the triangle in question be constant (say equal to $\frac{1}{8}S'$) the locus of T will be the cubic

$$abc\,S\Sigma-S'(S_1\alpha-S\alpha_1)(S_2\beta-S\beta_2)(S_3\gamma-S\gamma_3)=0\ldots\ldots\ldots(1),$$

which locus (it may easily be verified) passes through A″, B″, C″.

Now if the three points A″, B″, C″ lie on a right line $l\alpha+m\beta+n\gamma=0$, the coefficient of $\alpha\beta\gamma$ (as far as it appears explicitly) in Σ must vanish, and the remaining terms of Σ will have $l\alpha+m\beta+n\gamma$ as a common factor; in fact, Σ may then be written

$$(l\alpha+m\beta+n\gamma)\left\{\alpha_1(\beta_3\gamma_2)\frac{\beta\gamma}{l}+\beta_2(\alpha_3\gamma_1)\frac{\gamma\alpha}{m}+\gamma_3(\alpha_2\beta_1)\frac{\alpha\beta}{n}\right\};$$

and it at once appears that, for the value $S'=0$, the locus of T breaks up into the right line $l\alpha+m\beta+n\gamma=0$ and the conic $\alpha_1(\beta_3\gamma_2)\dfrac{\beta\gamma}{l}+\ldots=0$.

Finally, let the line $l\alpha+m\beta+n\gamma=0$ become the line at infinity, and the three points A″, B″, C″ be determined on it as the poles of those diameters of any conic $L\beta\gamma+M\gamma\alpha+N\alpha\beta=0$, circumscribing the triangle ABC, which are parallel to BC, AC, AB respectively, the area S' being again finite. It will

now be found that the locus of T is the line at infinity, and the conic

$$abc\left\{\frac{(\beta_3\gamma_2)}{\beta_2\gamma_3}\frac{\beta\gamma}{a}+\frac{(\alpha_3\gamma_1)}{\alpha_1\gamma_3}\frac{\gamma\alpha}{b}+\frac{(\alpha_2\beta_1)}{\alpha_1\beta_2}\frac{\alpha\beta}{c}\right\}-S'S^2=0\ldots\ldots(2),$$

where S is the area of the triangle ABC, since finite points only are considered. By forming the equations of the lines joining the points A, B, C with A″, B″, C″, it will be found that

$$\frac{\beta_1}{\alpha_1}=-\frac{a\mathrm{L}+b\mathrm{M}-c\mathrm{N}}{2b\mathrm{L}},\quad \frac{\alpha_2}{\beta_2}=-\frac{a\mathrm{L}+b\mathrm{M}-c\mathrm{N}}{2a\mathrm{M}},\quad \frac{\gamma_1}{\alpha_1}=\ldots;$$

and substituting these expressions in (2), the equation becomes

$$(a^2\mathrm{L}^2+b^2\mathrm{M}^2+c^2\mathrm{N}^2-2bc\mathrm{MN}-2ca\mathrm{NL}-2ab\mathrm{LM})(\mathrm{L}\beta\gamma+\mathrm{M}\gamma\alpha+\mathrm{N}\alpha\beta)$$
$$-4\mathrm{LMN}S'S^2=0,$$

a conic similar to and coaxal with $\mathrm{L}\beta\gamma+\mathrm{M}\gamma\alpha+\mathrm{N}\alpha\beta=0$ [Quest. 2464].

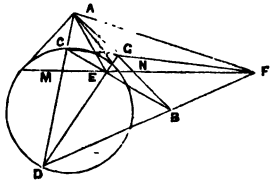
2383. (Proposed by W. K. Clifford, B.A.)—A and B are fixed points with regard to a conic, ACD a variable straight line passing through A and cutting the curve in C, D. The polar of A meets BC, BD in E, F; show that DE and CF meet in a fixed point G, and that ABG is a straight line.

Solution by J. Dale; W. H. Laverty, B.A.; *and others.*

Drawing the figure as directed, let M, N be the points in which the polar of A is cut by ACD and AB. Then

(E . ACMD) = (F . ACMD) = (F . ADMC);

therefore ED and FC intersect in the fixed line AB, and since the point of intersection G forms a harmonic range with the fixed points A, N, B, it is also fixed in position.



2551. (Proposed by Chief Justice Cockle, F.R.S.)—Given the partial differential equation $\frac{d^2z}{dy^2}-a^2\frac{d^2z}{dx^2}+\beta\frac{dz}{dy}+\gamma z=0\ldots\ldots\ldots\ldots\ldots(\mathrm{I})$,

wherein α, β, and γ are functions of x only. Show that when the condition

$$\gamma-\frac{1}{4}\left(\frac{da}{dx}+\beta\right)^2-\frac{a^2}{2}\frac{d}{dx}\left\{\frac{1}{a}\left(\frac{da}{dx}+\beta\right)\right\}=0\ldots\ldots\ldots(\mathrm{II})$$

is satisfied, then in general $z=\sqrt{a}\,\epsilon^{\int\frac{\beta}{2a}dx}\,\phi\left(y+\int\frac{dx}{a}\right)$,

where ϕ is an arbitrary function. Show also that, when β is constant and (II) satisfied, then the integral of (I) is

$$x = \sqrt{a}\, \epsilon^{\int \frac{\beta}{2a} dx} \phi\left(y + \int \frac{dx}{a}\right) + k\epsilon^{-\int \frac{\beta}{2a} dx} \psi\left(y - \int \frac{dx}{a}\right),$$

where ψ is another arbitrary function, and the arbitrary multiplier k is only inserted for convenience.

Solution by the PROPOSER.

From the equation $z = A\phi(y + B)$ ..... (1),

where ϕ is arbitrary and A and B are functions of x only, there results

$$\frac{d^2z}{dy^2} - \frac{1}{B'^2}\frac{d^2z}{dx^2} + \left(\frac{2A'B' + AB''}{AB'^2}\right)\frac{dz}{dy} + \frac{A''}{AB'^2}z = 0 \text{ ..... (2)},$$

which we may represent by $\frac{d^2z}{dy^2} - \alpha^2\frac{d^2z}{dx^2} + \beta\frac{dz}{dy} + \gamma z = 0$ ..... (3);

so that $\frac{1}{B'} = \alpha, \quad \frac{2A'B' + AB''}{AB'^2} = \beta, \quad \frac{A''}{AB'^2} = \gamma$ ..... (4, 5, 6).

Eliminating B′ by means of (4) from (5) and (6), we have

$$-\alpha'A + 2\alpha A' = \beta A, \quad \text{and} \quad \alpha^2 A'' = \gamma A \text{ ..... (7, 8)},$$

the accents throughout denoting differentiations with respect to x.

From (7) we deduce by differentiation

$$-\alpha''A + \alpha'A' + 2A''\alpha = \beta A' + \beta' A \text{ ..... (9)}.$$

Divide (9) by A, and by means of the sinister forms of (7) and (8) eliminate $\frac{A'}{A}$ and $\frac{A''}{A}$ from the quotient. We arrive at

$$-\alpha'' + \frac{\alpha'}{2\alpha}(\alpha' + \beta) + \frac{2\gamma}{\alpha} = \frac{\beta}{2\alpha}(\alpha' + \beta) + \beta' \text{ ..... (10)},$$

or $$\frac{2\gamma}{\alpha} + \frac{(\alpha' - \beta)(\alpha' + \beta)}{2\alpha} - \alpha'' - \beta' = 0 \text{ ..... (11)},$$

which reduces to $$\gamma + \tfrac{1}{4}(\alpha'^2 - \beta^2) - \tfrac{1}{2}(\alpha\alpha'' + \alpha\beta') = 0 \text{ ..... (12)},$$

whence $$\gamma - \tfrac{1}{4}(\alpha' + \beta)^2 + \tfrac{1}{2}(\alpha'^2 + \beta\alpha') - \tfrac{1}{2}(\alpha\alpha'' + \alpha\beta') = 0,$$

or lastly $$\gamma - \tfrac{1}{4}(\alpha' + \beta)^2 - \frac{\alpha^2}{2}\frac{d}{dx}\left(\frac{\alpha' + \beta}{\alpha}\right) = 0 \text{ ..... (13)}.$$

Whenever the condition (13) is fulfilled, (3) is satisfied by (1). Moreover B′, and consequently B, is obtained from (4), and we also have, by (5),

$$2\frac{A'}{A} + \frac{B''}{B'} = \beta B' \text{ ..... (14)},$$

whence $A^2B' = \epsilon^{\int \beta B' dx}$, or, by (4), $A^2 = \alpha\epsilon^{\int \frac{\beta}{\alpha} dx}$ ..... (15, 16).

Next, let $$z = A\phi(y + B) + C\psi(y - B) \text{ ..... (17)},$$

the functions ϕ and ψ being arbitrary, and A, B, and C being functions of x only. Also let $\frac{d^n\phi(y+B)}{d(y+B)^n} = \phi_n$, $\frac{d^n\psi(y-B)}{d(y-B)^n} = \psi_n$ .....(18, 19);

then we have $\frac{dz}{dy} = A\phi_1 + C\psi_1$, $\frac{d^2z}{dy^2} = A\phi_2 + C\psi_2$ .....(20, 21),

$$\frac{dz}{dx} = A\phi_1 B' + A'\phi - C\psi_1 B' + C'\psi \quad \text{.....(22)},$$

$$\frac{d^2z}{dx^2} = \left.\begin{array}{l} A\phi_2 B'^2 + (2A'B' + AB'')\phi_1 + A''\phi \\ + C\phi_2 B'^2 - (2C'B' + CB'')\psi_1 + C''\psi \end{array}\right\} \quad \text{.....(23)}.$$

Hence (17) will give rise to an equation of the form of (2) or (3), provided that

$$-\left(\frac{2C'B' + CB''}{CB'^2}\right) = \frac{2A'B' + AB''}{AB'^2} = \beta, \text{ and } \frac{C''}{CB'^2} = \frac{A''}{AB'^2} = \gamma \text{ ... (24, 25).}$$

But one branch of (24) gives $\frac{2C'B' + CB''}{CB'^2} + \frac{2A'B' + AB''}{AB'^2} = 0$ .....(26),

which, multiplied into B′ and reduced, leads, on division by (2), to

$\frac{C'}{C} + \frac{B''}{B'} + \frac{A'}{A} = 0$, whence, by integration, $ACB' = k$ ....(27, 28),

k being an arbitrary constant. Moreover one branch of (25) gives

$CA'' - AC'' = 0$, whence, by integration, $CA' - AC' = K$ ....(29, 30),

where K is another arbitrary constant. We have still, it will be remembered, from the second branch of (24) $\frac{2A'}{A} + \frac{B''}{B'} = \beta B'$ .....(14).

Hence, subtracting (27) from (14), $\frac{A'}{A} - \frac{C'}{C} = \beta B'$ .....(31).

But, dividing (30) by AC, we have $\frac{A'}{A} - \frac{C'}{C} = \frac{K}{AC}$ .....(32);

consequently, by (31) and (32), $\beta B' = \frac{K}{AC}$ .....(33);

but, by (28), $B' = \frac{k}{AC}$ .....(34);

hence, by (33) and (34) combined, $\beta = \frac{K}{k}$ .....(35).

The inference from this is, that when β is constant and (13) is satisfied, (3) has (17) for the form of its integral. Moreover, by (16),

$$A = \sqrt{a}\, e^{\int \frac{\beta}{2a} dx} \quad \text{.....(36)};$$

consequently by (34) and (4) $C = k\sqrt{a}\, e^{-\int \frac{\beta}{2a} dx}$ .....(37);

so that, when (13) is satisfied and β constant, the integral of (3) is

$$z = \sqrt{a}\, e^{\int \frac{\beta}{2a} dx} \phi\left(y + \int \frac{dx}{a}\right) + k\sqrt{a}\, e^{-\int \frac{\beta}{2a} dx} \psi\left(y - \int \frac{dx}{a}\right) \quad \text{....(38)}.$$

It may be well to test these results by an example. Given

$$\frac{d^2z}{dy^2} - \frac{d^2z}{dx^2} + 2\frac{dz}{dy} + z = 0.$$

Here $\alpha=1$, $\beta=2$, $\gamma=1$, the condition (13) is satisfied, and the solution derived from (38) is $z = \epsilon^x\phi(y+x) + k\epsilon^{-x}\psi(y-x)$,

which is verified as follows :—

$$\frac{dz}{dy} = \epsilon^x\phi_1 + k\epsilon^{-x}\psi_1,$$

$$\frac{d^2z}{dy^2} = \epsilon^x\phi_2 + k\epsilon^{-x}\psi_2,$$

$$\frac{dz}{dx} = \epsilon^x\phi_1 + \epsilon^x\phi - k\epsilon^{-x}\psi_1 - k\epsilon^{-x}\psi,$$

$$\frac{d^2z}{dx^2} = \epsilon^x\phi_2 + 2\epsilon^x\phi_1 + \epsilon^x\phi + k\epsilon^{-x}\psi_2 + 2k\epsilon^{-x}\psi_1 + k\epsilon^{-x}\psi,$$

whence $$\frac{d^2z}{dy^2} - \frac{d^2z}{dx^2} = -2(\epsilon^x\phi_1 + k\epsilon^{-x}\psi_1) - (\epsilon^x\phi + k\epsilon^{-x}\psi),$$

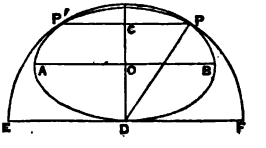
or $$\frac{d^2z}{dy^2} - \frac{d^2z}{dx^2} + 2\frac{dz}{dy} + z = 0,$$

as it ought.

2577. (Proposed by the EDITOR.)—Show that the average area of all the ellipses that can be drawn touching a semicircle and its diameter symmetrically, in parts of the area of the ellipse, is $\frac{2}{\pi}(2+\sqrt{2})$, or ·45523..., if (1) all possible positions of the centre of the ellipse be supposed equally probable, and $(2-\sqrt{2})$, or ·58579..., if (2) all distances of the chord of contact from the centre of the semicircle be supposed equally probable.

Solution by S. WATSON; *the* REV. J. WOLSTENHOLME, M.A.; *and others.*

1. It is plain the ellipse must always touch the diameter EF of the semicircle in its middle point D. Let O be the centre of the ellipse, and PCP′ the chord of contact with the semicircle. Put $ED=DP=r$, $DC=z$; and let $(2\alpha, 2\beta)$ be the axes of the ellipse, and (x, y) the coordinates of P relative to these axes. Then we have



$$r = \text{maximum value of } \left\{(\beta+y)^2 + x^2\right\}^{\frac{1}{2}}$$

$$= \text{maximum value of } \left\{(\beta+y)^2 + \alpha^2 - \frac{\alpha^2y^2}{\beta^2}\right\}^{\frac{1}{2}} = \frac{\alpha^2}{(\alpha^2-\beta^2)^{\frac{1}{2}}};$$

therefore $\beta^2 = a^2 - \frac{a^4}{r^2}$, and $\beta\, d\beta = \left(a - \frac{2a^3}{r^2}\right) da$.

Now, in (1), β may, with equal probability, take any value from 0 to $\frac{1}{2}r$; therefore $\frac{1}{2}r$ is the measure of the number of ellipses. Also the limits of a are r and $\frac{1}{2}r\sqrt{2}$; hence, in this case, the average area of the ellipse in parts of the semicircle is

$$\frac{2}{r}\int_0^{\frac{1}{2}r} \frac{a\beta d\beta}{\frac{1}{2}r^2} = \frac{4}{r^3}\int_r^{\frac{1}{2}r\sqrt{2}} \left(1 - \frac{2a^2}{r^2}\right) a^2 da = \frac{2}{15}(2+\sqrt{2}) = \cdot 45523\ldots$$

2. The value of y belonging to the maximum in (1) is

$$y = \frac{\beta^2}{a^2 - \beta^2}, \quad \text{therefore } z = \beta + y = \frac{a^2\beta}{a^2-\beta^2} = \frac{r(r^2-a^2)^{\frac{1}{2}}}{a}.$$

Also, since z may in this case, with equal probability, take any value from 0 to r, the measure of the number of ellipses is r; hence the average area of the ellipse in parts of the semicircle is

$$\frac{1}{r}\int_0^r \frac{a\beta dz}{\frac{1}{2}r^2} = -\frac{2}{r}\int_r^{\frac{1}{2}r\sqrt{2}} da = 2 - \sqrt{2} = \cdot 58579\ldots$$

[Referred to DF, DC, as axes of coordinates, the equation to a conic having double contact along the chord PP′ (or $y - z = 0$) with the circle $x^2 + y^2 - r^2 = 0$, that is the given semicircle APP′E, is

$$x^2 + y^2 - r^2 + k(y-z)^2 = 0,$$

and determining k so that the conic may touch DF at D (*i.e.*, the axis of x at the origin), we find $k = \frac{r^2}{z^2}$; thus the equation to the ellipse DPP′ is

$$x^2 + \left(1 + \frac{r^2}{z^2}\right) y^2 - \frac{2r^2}{z} y = 0, \quad \text{or} \quad \left(\frac{r^2+z^2}{r^4}\right) x^2 + \frac{(r^2+z^2)^2}{r^4 z^2}\left(y - \frac{r^2 z}{r^2+z^2}\right)^2 = 1,$$

and the semiaxes are $\alpha = \frac{r^2}{(r^2+z^2)^{\frac{1}{2}}}$, $\beta = \frac{r^2 z}{r^2+z^2}$;

whence we obtain $z = \frac{r}{a}(r^2-a^2)^{\frac{1}{2}}$, $\beta^2 = a^2 - \frac{a^4}{r^2}$,

as in the above solution. See also Hymers' *Conics*, p. 189.]

2738. (Proposed by R. Moon, M.A.)—The equation representing the transmission of sound parallel to the axis of a tube, when formed without reference to any previously assumed law of pressure, is $\frac{d^2y}{dt^2} + \frac{1}{D} \cdot \frac{dp}{dx} = 0$, where x is the ordinate of the point of rest of a particle, y the same ordinate at the time t, p the pressure at the same time, and D the density of equi-

librium. Show that the equation is satisfied by the values of the pressure, velocity, and density derived from the three following equations, viz.

$$p = -\frac{a^2}{\rho} + \chi\left(v + \frac{a}{\rho}\right), \qquad v + \frac{a}{\rho} = \phi\left(x + \frac{a-\chi'}{D}\, t\right),$$

$$\frac{1}{\rho} - f\left(v + \frac{a}{\rho}\right) = \psi\left(x - \frac{a}{D}\, t\right),$$

where a is an arbitrary constant; χ, ϕ, and ψ are arbitrary functions; and f is a function whose form is given by the equation

$$f(z) = \int \frac{dz}{2a - \chi'(z)}.$$

Solution by the PROPOSER.

The first equation of the solution gives

$$\frac{dp}{dx} = \frac{a^2}{\rho^2}\frac{d\rho}{dx} + \left\{\frac{dv}{dx} - \frac{a}{\rho^2}\frac{d\rho}{dx}\right\} \cdot \chi';$$

hence the differential equation of motion becomes

$$0 = \frac{dv}{dt} + \frac{\chi'}{D} \cdot \frac{dv}{dx} + \frac{a}{\rho^2} \cdot \frac{a-\chi'}{D} \cdot \frac{d\rho}{dx} \quad \ldots\ldots\ldots\ldots (1).$$

The second equation of the solution gives us

$$\frac{d}{dx}\left(v + \frac{a}{\rho}\right) = \phi' \cdot \left\{1 - \frac{t}{D} \cdot \chi'' \cdot \frac{d}{dx}\left(v + \frac{a}{\rho}\right)\right\},$$

$$\frac{d}{dt}\left(v + \frac{a}{\rho}\right) = \phi' \cdot \left\{\frac{a-\chi'}{D} - \frac{t}{D} \cdot \chi'' \cdot \frac{d}{dt}\left(v + \frac{a}{\rho}\right)\right\};$$

whence we have $0 = \frac{d}{dt}\left(v + \frac{a}{\rho}\right) - \frac{a-\chi'}{D} \cdot \frac{d}{dx}\left(v + \frac{a}{\rho}\right),$

or $$0 = \frac{dv}{dt} - \frac{a}{\rho^2}\frac{d\rho}{dt} - \frac{a-\chi'}{D}\left(\frac{dv}{dx} - \frac{a}{\rho^2}\frac{d\rho}{dx}\right) \quad \ldots\ldots\ldots\ldots (2).$$

But $$\frac{dv}{dx} = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{D}{\rho}\right) = -\frac{D}{\rho^2}\frac{d\rho}{dt};$$

and eliminating, by means of this, $\frac{d\rho}{dt}$ from (2), the latter becomes identical with (1). In like manner, the third equation of the solution gives

$$-\frac{1}{\rho^2}\frac{d\rho}{dx} + \frac{1}{\chi' - 2a} \cdot \frac{d}{dx}\left(v + \frac{a}{\rho}\right) = \psi',$$

$$-\frac{1}{\rho^2}\frac{d\rho}{dt} + \frac{1}{\chi' - 2a} \cdot \frac{d}{dt}\left(v + \frac{a}{\rho}\right) = -\frac{a}{D} \cdot \psi';$$

whence, eliminating ψ', we get

$$0 = -\frac{1}{\rho^2}\left(\frac{d\rho}{dt} + \frac{a}{D}\frac{d\rho}{dx}\right) + \frac{1}{\chi' - 2a} \cdot \left\{\frac{d}{dt}\left(v + \frac{a}{\rho}\right) + \frac{a}{D}\frac{d}{dx}\left(v + \frac{a}{\rho}\right)\right\}.$$

Eliminating $\frac{d\rho}{dt}$ as before, this last equation becomes identical with (1).

It thus appears that each of the two last equations of the proposed solution satisfies (1): and (1) is what the equation of motion becomes when the proposed value of p is substituted for it; *i.e.*, the relations between the pressure, velocity, density, ordinate, and time, above proposed, satisfy the equation of motion.

Moreover, not only is the equation of motion solved in the above manner with the requisite generality, but it is solved with a degree of generality greater than any one had supposed to be necessary, or conceived to be attainable.

On the other hand, no other solution of the complete equation of sound, possessing the necessary degree of generality, has ever been proposed; the most general hitherto obtained, that of Poisson, involving only one arbitrary function, which is universally admitted, and can be easily shown to be insufficient.

The cause of the failure of all attempts made heretofore towards obtaining the general solution, undoubtedly has been the vicious law of pressure, which, without proof or evidence, has been assumed in the discussion of the problem.

The absolutely groundless nature of this law, viz., the law expressed by the equation $p = a^2\rho$, may be shown in the clearest manner as follows:—

In any case of motion, the pressure must be capable of being expressed in terms of the ordinate and time. The same may be said of the density; the same of the velocity. Hence we are at liberty to assume

$$p = f_1(x, t), \quad \rho = f_2(x, t), \quad v = f_3(x, t).$$

From these three equations, we may eliminate two of the unknown quantities, as x and t; and so obtain a relation between the remaining three, viz., p, ρ and v. Solving the equation so obtained, we shall get $p = f(\rho, v)$. Now on what ground does the ordinary theory cut down this general equation to the particular equation $p = a^2\rho$? Simply on the ground that the latter formula expresses the pressure in the case of equilibrium.

How far this consideration is sufficient, may be seen as follows:—

In the case of equilibrium, we have

$$p = \text{funct. } x, \quad \rho = \text{funct. } x, \quad \text{whence } p = \text{funct. } \rho.$$

But we have no more reason to suppose that p is a function of ρ only in the case of motion, than we have to suppose that in the case of motion p is a function of x only; or ρ a function of x only; *i.e.*, we have no reason at all to suppose so, though the absurdity of such a supposition in the latter cases is palpable, while in the former case it is not immediately obvious.

With regard to the solution I have above put forth, I would observe, that it has been derived from an equation of motion formed irrespective of any theory as to the nature of the law of pressure; and it has been so derived, by assuming provisionally that the differential equation admits of *two* first integrals of the form given by Monge's method.

In order that that assumption may hold good, I have shown (see *Philosophical Magazine*, vol. xxxvi., p. 27) that *two* relations must obtain between p, ρ, and v, or their partial differential coefficients, both of which, by a remarkable coincidence, may be satisfied by a single relation between p, ρ, and v; viz., that given in the above solution. It may thus be seen that the fact of p being left disposable, instead of being arbitrarily determined beforehand, as in the received theory, is the cause of the success of the method which I have adopted; and that the assumption of the equation $p = a^2\rho$ is not only unwarrantable, but is positively mischievous.

I offer the solution set forth in the foregoing question, as possessing the requisite generality, and as the only one so endowed which has ever been obtained. Those who prefer adhering to the existing theory upon the subject, in addition to the other difficulties by which they are beset, must do two things, viz., first, they must show on physical grounds that the above solution is inadmissible; and secondly, they must elicit a solution of their own of adequate generality.

2743. (Proposed by M. JENKINS, M.A.)—Show that if p be a prime number and m and n any positive integers, the highest power of p contained in $\frac{\Pi(m+n)}{\Pi(m)\,\Pi(n)}$ may be obtained by expressing $m+n$ and either m or n in the scale of p; the number of times that it would be necessary to borrow in subtracting the latter number from the former being the index of the power of p required.

Solution by Professor CAYLEY.

1. In adding any two numbers, we carry a certain number of times; and it is easy to see that the sum of the digits of the two components, less the sum of the digits of the sum, is equal to nine times the number of carryings; moreover, that the number of carryings is equal to the number of borrowings, if either of the components be subtracted from the sum.

2. The same thing is true in any scale of notation, only, instead of nine, we have the radix of the scale, less unity: say the theorem is

$$S(m)+S(n)-S(m+n)=(p-1)x.$$

3. If p be a prime number, the number of times that the factor p occurs in $\Pi(m)$ is

$$E\left(\frac{m}{p}\right)+E\left(\frac{m}{p^2}\right)+E\left(\frac{m}{p^3}\right)+\&c.,$$

where $E\left(\frac{m}{p}\right)$ denotes the integer part of $\frac{m}{p}$, and similarly $E\left(\frac{m}{p^2}\right)$ &c. the integer part of $\frac{m}{p^2}$, &c.; the series is, of course, finite.

Hence the number of times that the factor p occurs in $\frac{\Pi(m+n)}{\Pi(m)\,\Pi(n)}$ is

$$N=E\left(\frac{m+n}{p}\right)+E\left(\frac{m+n}{p^2}\right)+\&c.-E\left(\frac{m}{p}\right)-E\left(\frac{m}{p^2}\right)-\&c.$$
$$-E\left(\frac{n}{p}\right)-E\left(\frac{n}{p^2}\right)-\&c.$$

4. Hence, expressing m, n, $m+n$ in the scale to the radix p, suppose

$$m=a+bp+cp^2+dp^3,\quad n=a'+b'p+c'p^2+d'p^3,\quad m+n=\alpha+\beta p+\gamma p^2+\delta p^3,$$

we have

$$E\left(\frac{m}{p}\right)+E\left(\frac{m}{p^2}\right)+\&c.=b+cp+dp^2+c+dp+d$$
$$=d(p^2+p+1)+c(p+1)+b;$$

and similarly for $\mathrm{E}\left(\frac{n}{p}\right)+$&c., $\mathrm{E}\left(\frac{m+n}{p}\right)+$&c.... ; whence

$$\begin{aligned}(p-1)\,\mathrm{N} = &\ \delta\,(p^3-1)+\gamma\,(p^2-1)+\beta\,(p-1)\\ &-d\,(p^3-1)-c\,(p^2-1)-b\,(p-1)\\ &-d'\,(p^3-1)-c'\,(p^2-1)-c'\,(p-1)\\ &=\left\{m+n-\mathrm{S}\,(m+n)\right\}-\left\{m-\mathrm{S}\,(m)\right\}-\left\{n-\mathrm{S}\,(n)\right\}\\ &=\mathrm{S}\,(m)+\mathrm{S}\,(n)-\mathrm{S}\,(m+n)=(p-1)\,x,\end{aligned}$$

if x be the number of times of carrying for the sum $m+n$, or of borrowing for the difference $(m+n)-m$ or $(m+n)-n$; that is, $\mathrm{N}=x$, the required theorem. I remark that although the foregoing expression of the number N is a very elegant and ingenious one, yet the original form of N, as given at the end of (3), is the natural and proper expression of the number of times that the factor p occurs in the binomial coefficient $\frac{\Pi\,(m+n)}{\Pi\,(m)\,\Pi\,(n)}$.

II. *Solution by the* Proposer.

If $\mathrm{I}\left(\frac{m}{p^r}\right)$ denote the greatest integer in $\frac{m}{p^r}$, the highest power of p in $\Pi\,(m)$ is $\mathrm{I}\left(\frac{m}{p}\right)+\mathrm{I}\left(\frac{m}{p^2}\right)+$&c.; and this may be shown to be equal to $\frac{m-\mathrm{S}_m}{p-1}$, where S_m denotes the sum of the digits of m expressed in the scale of p. Hence the required index is equal to $\frac{\mathrm{S}_m+\mathrm{S}_n-\mathrm{S}_{m+n}}{p-1}$, which evidently expresses the number of times it would be necessary to borrow in subtracting m (or n) from $m+n$; and this number can obviously be obtained by a simple inspection of the numbers m (or n) and $m+n$, expressed in the scale of p, without actually performing the subtraction.

2721. (Proposed by M. W. Crofton, F.R.S.)—Given any convex area Ω, let C be the length of any chord drawn across the area, and let the position of C be defined by (p, θ), the perpendicular on it from a fixed pole inside Ω and the inclination of p to a fixed axis; then $\iint \mathrm{C}^3\,dp\,d\theta = 3\Omega^2$, the integration extending to all possible positions of C.

Solution by the Proposer.

Consider the given area as covered with points uniformly distributed: and let us examine the infinite system of straight lines obtained by joining every pair of points. Call the pair of points A, B; and first, let A be fixed. Through A (Fig. 1) draw a chord PQ inclined at an angle θ to some fixed direction. Put $\mathrm{AP}=\rho$, $\mathrm{AQ}=\rho'$, $\mathrm{PQ}=\mathrm{C}$. Then,

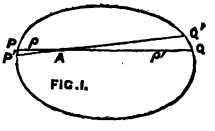


FIG. I.

drawing through A P′Q′, making an infinitesimal angle $d\theta$ with PQ, all the positions of second point B, which *give the direction of the line* AB *between the limits* θ *and* $\theta + d\theta$, are contained within the triangles PAP′, QAQ′; hence the number of such lines AB is measured by $\frac{1}{2}d\theta(\rho^2 + \rho'^2)$.

Hence, if dS be an element of the area at A, and if A now be not fixed, but *taken anywhere within* dS, the number of lines AB between the directions θ and $\theta + d\theta$ will be $\frac{1}{2}d\theta(\rho^2 + \rho'^2)\,dS$.

Consequently, if A be anywhere within the space between PQ and a parallel chord, distant dp from it (Fig. 2), the number of lines AB between θ and $\theta + d\theta$ will be

$$\tfrac{1}{2}d\theta\,dp\int_0^C \left\{\rho^2 + (C-\rho)^2\right\}d\rho = \tfrac{1}{3}C^3\,dp\,d\theta.$$

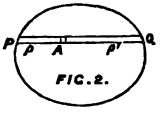


FIG. 2.

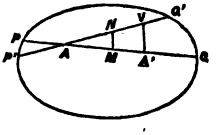
Now if A be *anywhere within the boundary*, the number of lines AB between the directions θ and $\theta + d\theta$, is $\frac{1}{3}d\theta\int C^3\,dp$, the chord C always having the direction θ. Hence, allowing θ also to vary, we have, for the total number of lines AB, $N = \frac{1}{3}\iint C^3\,dp\,d\theta$. But we know the total number to be Ω^2. Hence $\iint C^3\,dp\,d\theta = 3\Omega^2$.

NOTE.—Hence, if L be the perimeter of any convex area Ω, the average value of the cube of a chord drawn at random across it is $\frac{3\Omega^2}{L}$. The average value of the chord itself is $\frac{\pi\Omega}{L}$. Hence *the average length of a line drawn at random across a triangle is the same as that of one drawn at random across the inscribed circle, viz., a quarter of the circumference of that circle.* The same is true for any polygon circumscribed about a circle.

2680. (Proposed by Capt. CLARKE, R.E., F.R.S.)—If A and B be two points taken at random within a closed convex figure, and the line AB be produced to meet the boundary in P and Q, prove that the chance that the three segments PA, AB, BQ are such as would form a triangle is $\frac{1}{4}$.

Solution by M. W. CROFTON, F.R.S.

A solution of this question may be readily deduced from that of the preceding Question 2721. For let A be a fixed point within the boundary, and B a second point taken at random within it. Through A draw two chords PQ, P′Q′, whose directions are θ and $\theta + d\theta$. Confining our attention at present to lines AB between θ and $\theta + d\theta$, let us examine the favourable positions of B (that is, those which render it possible to form a triangle with the three segments of the line AB produced both ways to meet the boundary). Now if we bisect PQ in M, and take MA′ = AP, and draw MN, A′V perpendicular to PQ, it will be seen that all the favourable



positions of B are within the quadrilateral MNVA'. Hence the number of favourable cases is represented by

$$F = \tfrac{1}{2}d\theta\left\{\tfrac{1}{4}(\rho+\rho')^2-\tfrac{1}{4}(\rho-\rho')^2\right\} = \tfrac{1}{2}\rho\rho' d\theta.$$

As in the solution of Quest. 2721, suppose A now anywhere between PQ and a parallel chord, distant dp from it; then the number of cases will be

$$F = \tfrac{1}{2}d\theta\, dp \int_0^C \rho\,(C-\rho)\,d\rho = \tfrac{1}{12}C^3 dp\, d\theta.$$

Finally, the whole number of favourable cases will be

$$F = \tfrac{1}{12}\iint C^3 dp\, d\theta = \tfrac{1}{4}\Omega^2.$$

Hence, as the whole number of cases is Ω^2, the theorem is proved.

By the same method the following theorem may be proved:—

Two points A, B *are taken at random within any convex boundary; if the line* AB *be produced to meet the boundary in* P *and* Q, *the chance that* AB *is less than half* PQ *is* $\frac{1}{2}$.

2579. (Proposed by T. COTTERILL, M.A.)—1. If a line and circle are given, and Q is a point on the tangent to the circle at P so that PQ is bisected by the line, the locus of Q is a quartic with three double points, each branch having an inflexion as it passes through a double point. If the line touches the circle, the quartic becomes the line and a Cissoid of Diocles.

2. Reciprocally, find the envelope of the chord of intersection of a central conic with its circles of curvature. What is it in the parabola?

I. *Solution by the* REV. J. WOLSTENHOLME, M.A.

1. If $x^2+y^2=a^2$ be the circle, $x=h$ the straight line, $x\cos\theta + y\sin\theta = a$ the tangent at P, and (X, Y) the coordinates of Q; then $X+a\cos\theta = 2h$, $Y+a\sin\theta = 2(a-h\cos\theta)\operatorname{cosec}\theta$, and the locus is

$$Y^2\left\{a^2-(X-2h)^2\right\} = (a^2-2hX+X^2)^2,$$

which becomes $(X-a)\left\{Y^2(3a-x)-(X-a)^3\right\} = 0$,

when $h=a$. If $h>a$, the real branches of the curve are as represented in Fig. 1. If $h<a$, the curve does not meet the axis of x in any real point, but touches the circle at its intersections with the given straight line and line at infinity.

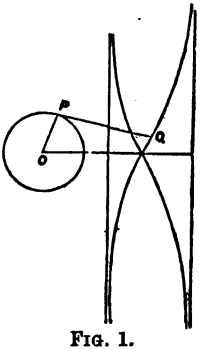


FIG. 1.

2. The equation of the chord of intersection of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its circle of curvature at the point $(a\cos\theta, b\sin\theta)$ is $\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta = \cos 2\theta$.

For envelope $\frac{x}{a}\sin\theta + \frac{y}{b}\cos\theta = 2\sin 2\theta$;

therefore $\frac{x}{a} = \cos\theta\cos 2\theta + 2\sin\theta\sin 2\theta = \cos^3\theta + 3\sin^2\theta\cos\theta,$

and $\frac{y}{b} = 2\cos\theta\sin 2\theta - \sin\theta\cos 2\theta = \sin^3\theta + 3\sin\theta\cos^2\theta;$

hence we have $\frac{x}{a} \pm \frac{y}{b} = (\cos\theta \pm \sin\theta)^3;$

and the equation of the envelope is

$$\left(\frac{x}{a} + \frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{x}{a} - \frac{y}{b}\right)^{\frac{2}{3}} = 2,$$

a curve having cusps (as in Fig. 2) on the equiconjugate diameters at the points $\frac{x^2}{a^2} = \frac{y^2}{b^2} = 2.$

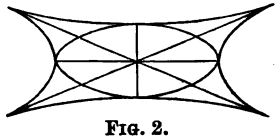


FIG. 2.

In the parabola $y^2 = 4ax$, the equation of the chord of intersection of the curve with the circle of curvature at $(am^2, 2am)$ is

$$y - 2am = -\frac{1}{m}(x - am^2), \quad \text{or} \quad 3am - y - \frac{x}{m} = 0,$$

whose envelope is $y^2 + 12ax = 0.$

II. *Solution by the* PROPOSER.

If the three pairs of lines through four points meet in X, Y, Z, then a line in the plane is touched by two conics through the four points in two points, harmonic conjugates to the conics of the system, including the three pairs of lines through them. The points are inverse to the triangle XYZ, their coordinates being connected by equations of the form $\alpha x x' = b y y' = c z z'$. The inverse to a conic of the system is easily found by taking on a tangent to the conic the harmonic conjugate to any pair of lines through the four points. The equation to the conic being of the form $\alpha x^2 + \beta y^2 + \gamma z^2 = 0$, the inverse curve is of the form $\alpha' x^2 + \beta' y^2 + \gamma' z^2 = 0$. A simple case of this is the problem (1).

If a conic is inscribed in four lines, the envelope of the line which, with a tangent to the conic, forms a pair of lines conjugate to the quadrilinear, is the reciprocal of the last mentioned locus. If the four lines intersect in the foci of the curve, the envelope is the evolute of the curve, and a full discussion of its properties is given by Prof. CREMONA in the *Messenger of Mathematics*, Vol. III., p. 88. It will readily be seen that, in (2), the circumscribing rectangle parallel to the axes forms the quadrilinear.

2733. (Proposed by the EDITOR.)—Show that the average area of an infinite series of triangles, equably distributed over the surface of an ellipse so as to have each one side parallel to a given line, in parts of the area of the ellipse, is $\frac{256}{525\pi^2}$.

I. *Solution by* THOMAS SAVAGE, M.A.

1. Imagine a series of chords of a *circle* drawn at very small equal distances parallel to a given line. Then we may consider the area of the circle to be filled with points uniformly distributed along these chords, the number of points in each chord being of course proportional to the length of the chord.

Let PQR be the triangle. The number of triangles in which P, Q fall on a chord at a distance x from the centre, is proportional to the square of the length of the chord, or to r^2-x^2, if r be the radius. The number of triangles in which R falls on a chord at a distance y from the centre, is proportional to $(r^2-y^2)^{\frac{1}{2}}$, and the average area of the triangles thus formed is

$$\tfrac{1}{2}\,(\text{average length of PQ})\,(y \curvearrowleft x).$$

Now, the average length of PQ is $\frac{\pi}{4}(r^2-x^2)^{\frac{1}{2}}$ as is easily seen. Hence, putting (Δ) for the average area of the triangle, PQR, we have

$$(\Delta) = \frac{\iint (r^2-x^2)\,(r^2-y^2)^{\frac{1}{2}}\,\frac{1}{2}\,(r^2-x^2)^{\frac{1}{2}}\,(y \curvearrowleft x)\,dx\,dy}{\iint (r^2-x^2)\,(r^2-y^2)^{\frac{1}{2}}\,dx\,dy},$$

where $y \curvearrowleft x$ is positive throughout the integration. Put $x = r\cos\alpha$, $y = r\cos\beta$, and the integrals in the value of (Δ) reduce to

$$\frac{\frac{r^2}{3}\cdot\int_0^{\pi}\sin^4\alpha\left\{\int_0^{\alpha}\sin^2\beta\,(\cos\beta-\cos\alpha)\,d\beta+\int_{\alpha}^{\pi}\sin^2\beta\,(\cos\alpha-\cos\beta)\,d\beta\right\}d\alpha}{\int_0^{\pi}\int_0^{\pi}\sin^3\alpha\,\sin^2\beta\,d\alpha\,d\beta}$$

Now $\displaystyle\int_0^{\alpha}\sin^2\beta\cos\beta\,d\beta = \frac{\sin^3\alpha}{3}$, $\displaystyle\int_{\alpha}^{\pi}\sin^2\beta\cos\beta\,d\beta = -\frac{\sin^3\alpha}{3}$,

$$\int_0^{\alpha}\sin^2\beta\,d\beta = \frac{\alpha}{2}-\frac{\sin\alpha\cos\alpha}{2},\quad \int_{\alpha}^{\pi}\sin^2\beta\,d\beta = \frac{\pi-\alpha}{2}+\frac{\sin\alpha\cos\alpha}{2}.$$

Substituting these values, we find that the average area of the triangle PQR is

$$(\Delta) = \frac{r^2}{2\pi}\int_0^{\pi}\sin^4\alpha\left\{\frac{2}{3}\sin^3\alpha+\cos\alpha\left(\tfrac{1}{2}\pi-\alpha+\sin\alpha\cos\alpha\right)\right\}d\alpha$$

$$= \frac{r^2}{2\pi}\int_0^{\pi}\left(\frac{6}{5}\sin^5\alpha-\frac{1}{3}\sin^7\alpha\right)d\alpha,$$

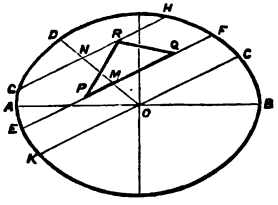
since $\displaystyle\int_0^{\pi}\sin^4\alpha\cos\alpha\,d\alpha = 0$, and $\displaystyle\int_0^{\pi}\sin^4\alpha\cos\alpha\,\alpha\,d\alpha = -\tfrac{1}{2}\int_0^{\pi}\sin^5\alpha\,d\alpha$.

Hence, finally, the average area of the triangle PQR is

$$(\Delta) = \frac{r^2}{\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{6}{5}\sin^5\alpha-\frac{1}{3}\sin^7\alpha\right) = \frac{r^2}{\pi}\left(\frac{6}{5}\cdot\frac{4.2}{5.3}-\frac{1}{3}\cdot\frac{6.4.2}{7.5.3}\right) = \frac{256r^2}{525\pi}.$$

2. The solution to the problem is the same as the above, when the triangle

PQR is drawn on the surface of any *ellipse*. For suppose, in the accompanying figure, the triangles, of which PQR is one, to have each one side parallel to the diameter COK of the ellipse ADB, OD being the semi-diameter conjugate to OC; then, by precisely the same method as that already employed, we obtain as the average area of all the triangles



$$(\Delta) = \frac{256}{525\pi}\, OC \,.\, OD \sin COD.$$

But the area of the ellipse is

$$\pi \,.\, OC \,.\, OD \sin COD;$$

hence, whatever be the direction of the given line to which COK and PQ are to be parallel, the average area of the triangle PQR, in parts of the area of the ellipse, is $\frac{256}{525\pi^2}$, or ·0494..., that is $\frac{1}{20}$ nearly.

[The value of (Δ) when the three points P, Q, R may, with equal probability, take *any positions* within the ellipse, in parts of the area of the ellipse, has been shown by Mr. WOOLHOUSE and others to be $\frac{35}{48\pi^2}$, or ·07388. See *Reprint*, Vol. VIII., pp. 88 and 105, and Vol. IX., p. 97.]

II. *Solution by* STEPHEN WATSON.

Let CK be a diameter parallel to the given line (See Fig. to preceding Solution), and OD its conjugate; also draw the chords EF, GH, parallel to CK, and cutting OD in M, N. Let a, b be the semi-axes of the ellipse, $OC = a_1$, $OD = b_1$, $OM = y$, $ON = z$, and $\angle DOC = \phi$; then we have

$$EF = \frac{2a_1}{b_1}(b_1^2 - y^2)^{\frac{1}{2}}, \qquad GH = \frac{2a_1}{b_1}(b_1^2 - z^2)^{\frac{1}{2}}.$$

Now if two points P and Q take every position on EF, the sum of the distances PQ is $\frac{1}{6}EF^3$; hence, EF remaining fixed and R being a point on GH, the sum of the areas of the triangle PQR, while R takes every position in the elliptic segment EDF, is

$$\int_y^{b_1} \tfrac{1}{6} EF^3 (z-y) \sin\phi \,.\, GH \,.\, d(z \sin\phi) = \frac{a_1}{3b_1} EF^3 \sin^2\phi \int_y^{b_1} (z-y)(b_1^2 - z^2)^{\frac{1}{2}} dz$$

$$= \frac{a_1}{3b_1} EF^3 \sin^2\phi \left\{ (\tfrac{1}{3}b_1^2 + \tfrac{1}{8}y^2)(b_1^2 - y^2)^{\frac{1}{2}} + \tfrac{1}{2}b_1^2 y \left(\sin^{-1}\frac{y}{b_1} - \frac{\pi}{2} \right) \right\} \ldots (1).$$

Similarly, when R lies in the segment EBF, the sum of the areas is

$$\frac{a_1}{3b_1} EF^3 \sin^2\phi \left\{ (\tfrac{1}{3}b_1^2 + \tfrac{1}{8}y^2)(b_1^2 - y^2)^{\frac{1}{2}} + \tfrac{1}{2}b_1^2 y \left(\sin^{-1}\frac{y}{b_1} + \frac{\pi}{2} \right) \right\} \ldots . (2).$$

Hence, when R takes every position in the ellipse, and EF every position parallel to CK, the sum of the areas is

$$2\int_0^{b_1}\{(1)+(2)\}\,d\,(y\sin\phi)$$

$$=\frac{32a_1^4\sin^3\phi}{3b_1^4}\int_0^{b_1}\left\{(\tfrac{1}{8}b_1^2+\tfrac{1}{8}y^2)(b_1^2-y^2)^{\frac{1}{2}}+\tfrac{1}{8}b_1y(b_1^2-y^2)^{\frac{3}{2}}\sin^{-1}\frac{y}{b_1}\right\}dy$$

$$=\frac{4096}{1575}a_1^4b_1^3\sin^3\phi=\frac{4096}{1575}a^2b^2a_1^2b_1\sin\phi\quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

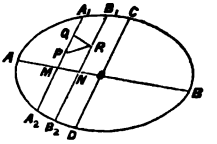
The number of triangles formed is

$$2ab\pi\int_0^{b_1}\mathrm{EF}^2.\,d\,(y\sin\phi)=\frac{16}{3}ab\,a_1^2b_1\pi\sin\phi\,;$$

therefore the required average $=\dfrac{256ab}{525\pi}=\dfrac{256}{525\pi^2}$ (area of ellipse),

which is independent of the direction of the given line.

III. *Solution by* Samuel Bills.



In the accompanying diagram, let ABCD be an ellipse, O being the centre, CD a diameter to which one of the sides of each of the triangles is to be parallel, and AB conjugate to CD. Draw A_1A_2, B_1B_2 parallel to CD, and let PQR be one of the triangles. Let A_1A_2, B_1B_2 respectively intersect AO in M and N. Put $AO=a_1$, $CO=b_1$, $OM=a_1\cos\theta_1$, $A_1M=b_1\sin\theta_1$, $ON=a_1\cos\theta_2$, $B_1N=b_1\sin\theta_2$, and angle $AOC=\alpha$. For shortness, put s_1 for the sine and c_1 for the cosine of θ_1; s_2, c_2 for the sine and cosine of θ_2; and s for the sine of α. Let $A_1P=u$ and $A_1Q=v$. Now when Q is *above* P we shall have area of $PQR=\frac{1}{2}a_1s\,(u-v)\,(c_1-c_2)$; and when Q is *below* P the area will $=\frac{1}{2}a_1s\,(v-u)\,(c_1-c_2)$. Hence the sum of the areas, while Q ranges from A_1 to A_2, will be

$$=\tfrac{1}{2}a_1(c_1-c_2)s\left\{\int_0^u(u-v)\,dv+\int_u^{2b_1s_1}(v-u)\,dv\right\}$$

$$=a_1(c_1-c_2)s\,(b_1^2s_1^2-b_1s_1u+\tfrac{1}{2}u^2).$$

And while P also ranges from A_1 to A_2, the sum of the areas will be

$$a_1(c_1-c_2)s\int_0^{2b_1s_1}(b_1^2s_1^2-b_1s_1u+\tfrac{1}{2}u^2)\,du=\tfrac{4}{3}(c_1-c_2)\,a_1b_1^3ss_1^3;$$

and if R also ranges from B_1 to B_2, the sum of the areas is

$$\tfrac{4}{3}a_1b_1^4ss_1^3s_2(c_1-c_2).$$

Hence, if Σ denote the total sum of the areas for the whole ellipse, we have

$$\Sigma=\frac{16}{3}a_1^3b_1^4s\int_0^{\pi}\int_0^{\theta_2}s_1^3s_2(c_1-c_2)\,d\,(1-c_2)\,d\,(1-c_1)$$

$$=\frac{16}{3}a_1^3b_1^4s\int_0^{\pi}\int_0^{\theta_2}s_1^4s_2^2(c_1-c_2)\,d\theta_2\,d\theta_1=\frac{16}{3}\cdot\frac{256}{525}a_1^3b_1^4s\quad\ldots.\ (1).*$$

Again; to find the number N of triangles. The number, while P and Q each range from A_1 to A_2, and R ranges from B_1 to B_2, will evidently be $8b_1^3 s_1^2 s_2$; hence the total number is

$$N = 16a_1^2 b_1^3 \int_0^{\pi} \int_0^{\theta_2} s_1^2 s_2 \, d(1-c_2)\, d(1-c_1)$$

$$= 16a_1^2 b_1^3 \int_0^{\pi} \int_0^{\theta_2} s_1^3 s_2^2 \, d\theta_2 d\theta_1 = \frac{16}{3} \pi a_1^2 b_1^3 \ldots\ldots\ldots\ldots (2).*$$

Hence the required average is

$$\frac{\Sigma}{N} = \frac{16 . 256 a_1^3 b_1^4 s}{3 . 525} \div \frac{16\pi a_1^2 b_1^3}{3} = \frac{256 a_1 b_1 s}{525\pi} = \frac{256ab}{525\pi},$$

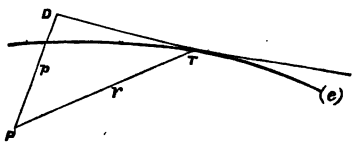
where a, b are the semi-axes of the ellipse.

* (1) and (2) were each doubled because B_1B_2 may be on *either* side of A_1A_2.

2732. (Proposed by W. K. CLIFFORD, B.A.)—An epi- or hypo-cycloid is pushed through a very short fixed tube, so as to remain in one plane, show that the locus of its centre is an ellipse.

Solution by W. S. BURNSIDE, M.A.

In general, if $r = fp$, where r is the radius vector and p the length of the perpendicular on the tangent, be the equation of the given curve, the locus of the pole P is given by the equation $(x^2+y^2)^{\frac{1}{2}} = f(x)$, when the curve (C) always touches the tangent TD at the point T.



In this particular case, $r^2 = a^2 + \frac{4m}{(m+1)^2} p^2$

(Salmon's *Higher Plane Curves*, Art. 227); whence the locus of the centre of the hypocycloid, which is the pole now, is given by the equation

$$(m+1)^2 (x^2+y^2) - 4mx^2 = (m+1)^2 a^2, \text{ or } \frac{x^2}{(m+1)^2} + \frac{y^2}{(m-1)^2} = b^2,$$

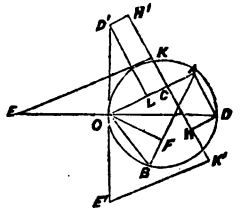
as $a+b = mb$, a being the radius of the fixed circle, and b the radius of the moving circle, in the description of the hypocycloid.

2100. (Proposed by the EDITOR.)—1. From one end O of a fixed diameter OD of a circle, two chords OA and OB are drawn such that the

rectangle OA . OB = a constant (k^2); show that the envelope of the line AB is a circle with centre O and radius a third proportional to OD and k.

2. On a chord OA of a circle OAD, a point C is taken such that the difference or (3) the sum of the squares on OC and OA = a constant (k^2); show that the envelope of the perpendicular drawn to OC through C is a conic having O as centre. [From the *Nouvelles Annales de Mathématiques.*]

I. *Solution by* MORGAN JENKINS, B.A.



1. Draw OF perpendicular to AB, produced if necessary; then the triangles OFB, OAD are similar, and OF : OB = OA : OD; hence OF is constant, and therefore the envelope is a circle with centre O [and radius a third proportional to OD and k, since OF . OD = k^2.]

2. Produce DO to E, making OE = OD; draw DH, EK parallel to OA, and through C draw KCH perpendicular to OA and meeting DH, EK in H, K respectively. Then DH . EK = the difference of the squares on OC and OA = k^2; therefore the envelope of KCH is in this case a central conic with D and E as foci, the conic being an ellipse or hyperbola according as OC is greater or less than OA.

3. When the *sum* of the squares on OA and OC = the constant k^2, draw D'OE' a tangent to the circle making OD' = OE' = OD, and draw D'L, D'H', perpendicular to OA, HK.

Then from similar triangles D'OL, ODA, we have OL = AD, and

$$D'H' \cdot E'K' = OC^2 \sim OL^2 = OC^2 \sim AD^2 = OA^2 + OC^2 \sim OD^2 = k^2 \sim OD^2.$$

And it will be seen that the perpendiculars D'H', E'K' are on the same or opposite sides of H'K' according as k > or < OD. Hence the envelope of H'K' is an ellipse or hyperbola with D', E' for foci according as k is > or < OD. When k = OD, the envelope reduces to the two points D', E'; and this is at once seen to be true by observing that the perpendicular from either of these points on OC forms with OC and D'OE' a triangle equal in all respects to the triangle OAD.

II. *Solution by* J. J. WALKER, M.A.

1. The envelope of the secant AB is obviously a circle with O as centre, since in any triangle the length of the perpendicular on the base is the fourth proportional to the diameter of the circumscribing circle and the two sides [in this case a third proportional to OD and k].

2. Let OD = d, and $\angle AOD = \phi$; and let (ρ, θ) be the polar coordinates of a point in HCK, referred to O as pole and OD as polar axis. Then we have

$$\rho^2 \cos^2(\phi - \theta) = OC^2 = OA^2 \mp k^2 = d^2 \cos^2\phi \mp k^2,$$

or

$$(\rho^2 \sin^2\theta \pm k^2) \tan^2\phi + \rho^2 \sin 2\theta \tan\phi + \rho^2 \cos^2\theta - d^2 \pm k^2 = 0 \ldots\ldots (1).$$

Differentiating with respect to the parameter $\tan\phi$, we have

$$2(\rho^2 \sin^2\theta \pm k^2) \tan\phi + \rho^2 \sin 2\theta = 0 \ldots\ldots\ldots\ldots (2).$$

Finally, eliminating $\tan\phi$ between (1) and (2), we have

$$\rho^2 \left\{ k^2 \cos^2\theta \mp (d^2 \mp k^2) \sin^2\theta \right\} = (d^2 \mp k^2) k^2 \ldots\ldots\ldots\ldots (3)$$

as the equation to the required envelope (the upper or lower sign being used according as OC is taken less or greater than OA); and it is plainly that of a central conic, having O as its centre and D as one of its foci; and being a hyperbola if OC be taken less than OA, an ellipse if otherwise.

[A slight variation of Mr. WALKER'S method may be obtained by writing the equation of HCK in the form (taking the case when the *sum* of the squares on OA and OC $= k^2$) $(x\cos\phi + y\sin\phi)^2 + d^2\cos^2\phi = k^2$,

and, putting $\tan\phi = t$, this equation becomes

$$(x+yt)^2 - k^2t^2 \equiv \{x+(y+k)t\}\{x+(y-k)t\} = k^2-d^2 \ldots\ldots (\alpha).$$

Differentiating (α) with respect to t, we have

$$t = \frac{xy}{k^2-y^2}, \quad x+(y+k)t = \frac{kx}{k-y}, \quad x+(y-k)t = \frac{kx}{k+y};$$

hence, substituting in (α), the equation of the envelope is found to be

$$\frac{x^2}{k^2-d^2} + \frac{y^2}{k^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (\beta),$$

a conic (ellipse or hyperbola according as $k >$ or $< d$) having O as centre, and D', E' as foci.]

2756. (Proposed by J. GRIFFITHS, M.A.)—Show that an infinite number of triangles can be described such that each has the same circumscribing, nine-point, and self-conjugate circles as a given triangle.

Solution by Professor CAYLEY.

It is a known theorem that if two triads of points, say A, B, C and A', B', C', are self-conjugate in regard to a conic S, they lie in a conic Σ. Take the conic S and the points A, B, C as given; then Σ will be a conic passing through A, B, C; and if on this conic we take any point A', and then take B' to be either of the intersections of the conic Σ by the polar of A in regard to S, and finally construct C' as the pole of A'B' in regard to S, then, by what precedes, C' will be on a conic through A, B, C, A', B', that is, on the conic Σ. Or given the conic S, the triangle ABC, and the conic Σ through A, B, C, we obtain an infinity of triangles A'B'C', self-conjugate in regard to S and inscribed in Σ. That is, if S, Σ are circles, we have an infinity of triangles self-conjugate in regard to the circle S and inscribed in the circle Σ; and inasmuch as the nine-points circle can be constructed by means of the two circles S, Σ alone, the triangles have all of them the same nine-points circle.

2334. (Proposed by W. S. BURNSIDE, M.A.)—1. Construct a triangle of given species, having its vertices on three concentric circles.

2. The triangle being equilateral, what relation exists between the radii when only one such triangle can be so constructed?

I. *Solution by* W. S. McCay, B.A.

1. The problem is the same as to find a point the ratios of whose distances from the vertices of a triangle are given; for then the figure may be enlarged to scale. Now when we have given the base and ratio of the sides of a triangle, the locus of the vertex is a circle drawn on the segment of the base that cuts it harmonically in the given ratio; hence the required point must lie on three such circles. But since the ratios of section of the three sides are connected (from the nature of the question) by the relation $l \cdot m \cdot n = 1$, therefore three of the points of section connect through a point to the opposite vertices, and the other three lie on a right line; hence the three segments may be considered as the diagonals in magnitude and position of a complete quadrilateral; and we know that the three circles described on these have a common radical axis; so there are, in general, *two* solutions to the question.

2. If there were only one solution, the point must lie on the circumscribing circle of the triangle; for it is well known that this circle cuts the other three orthogonally; therefore, if the three circles touch each other, the circumscribing circle must go through the point of contact. Hence, if α, β, γ be the radii of the given circles, and $a : b : c$ be the given ratio of sides of the triangle, Ptolemy's theorem gives the relation $a\alpha \pm b\beta \pm c\gamma = 0$.

II. *Solution by the* Proposer.

1. Let the radii of the three circles be denoted by α, β, γ, and the sides of the required triangles by $\lambda a, \lambda b, \lambda c$; then, making use of the relation which exists between the six connectors of four points in a plane, we have

$$\lambda^4 a^2b^2c^2 + \lambda^2 \begin{vmatrix} a^2\alpha^2 & a^2(a^2-b^2-c^2) \\ b^2\beta^2 & b^2(b^2-c^2-a^2) \\ c^2\gamma^2 & c^2(c^2-a^2-b^2) \end{vmatrix} + \begin{vmatrix} a^2(a^2-\beta^2) & (a^2-\gamma^2) \\ b^2(\beta^2-a^2) & (\beta^2-\gamma^2) \\ c^2(\gamma^2-a^2) & (\gamma^2-\beta^2) \end{vmatrix} = 0,$$

an equation which in general gives two distinct values for λ, and therefore two solutions of the question.

2. The discriminant of the above equation is found to be

$$M(b\beta + c\gamma - a\alpha)(c\gamma + a\alpha - b\beta)(a\alpha + b\beta - c\gamma),$$

where $$M = (b+c-a)(c+a-b)(a+b-c),$$

which vanishes when $a\alpha \pm b\beta \pm c\gamma = 0$, the geometric interpretation of which is obvious. Hence, if $a = b = c$, we have $\alpha \pm \beta \pm \gamma = 0$, the required relation between the radii.

2474. (Proposed by Professor Everett.)—

1. If $\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 = 0$, show that, when n is any positive integer,

$$\left(\cos\theta \cdot \frac{d}{dx} + \sin\theta \cdot \frac{d}{dy}\right)^n = \left(\cos n\theta \cdot \frac{d}{dx} + \sin n\theta \cdot \frac{d}{dy}\right)\left(\frac{d}{dx}\right)^{n-1};$$

and hence (2) prove the theorem (due to Rankine and Stokes) that at every multiple point in a plane curve which fulfils the condition $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$, the branches make equal angles with each other.

Solution by the PROPOSER.

1. If $a^2+b^2=0$, we have $b=ia$ [where $i=\sqrt{(-1)}$], and

$$(a\cos\theta+b\sin\theta)^n = a^n(\cos\theta+i\sin\theta)^n = a^n(\cos n\theta+i\sin n\theta)$$
$$= a^{n-1}(a\cos n\theta+b\sin n\theta).$$

Hence, writing $\frac{d}{dx}$ for a and $\frac{d}{dy}$ for b, the required equation is proved.

It may also be proved directly without the use of imaginaries, as follows:

$\left(\cos A\cdot\frac{d}{dx}+\sin A\cdot\frac{d}{dy}\right)\left(\cos B\cdot\frac{d}{dx}+\sin B\cdot\frac{d}{dy}\right)$ is, by actual development,

$$=\left\{\cos(A+B)\frac{d}{dx}+\sin(A+B)\frac{d}{dy}\right\}\frac{d}{dx}, \quad \text{since } \left(\frac{d}{dy}\right)^2=-\left(\frac{d}{dx}\right)^2.$$

If $A=B=\theta$, this becomes

$$\left(\cos\theta\cdot\frac{d}{dx}+\sin\theta\cdot\frac{d}{dy}\right)^2=\left(\cos 2\theta\cdot\frac{d}{dx}+\sin 2\theta\cdot\frac{d}{dy}\right)\frac{d}{dx}.$$

Hence the equation holds when $n=2$, and by putting $A=(m-1)\theta$, $B=\theta$, we may show that, if it holds for any one integral value of n, it holds for the next greater.

2. To apply this result to the proof of Rankine and Stokes' theorem. At a point where n branches of any curve $\phi(x,y)=0$ cut each other, the condition $\left(dx\frac{d}{dx}+dy\frac{d}{dy}\right)^n\phi=0$ is fulfilled, which may also be written $\left(\cos\theta\cdot\frac{d}{dx}+\sin\theta\cdot\frac{d}{dy}\right)^n\phi=0$, θ denoting the inclination of any branch to the axis of x. When $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0$, this condition reduces to

$$\left(\cos n\theta\cdot\frac{d}{dx}+\sin n\theta\cdot\frac{d}{dy}\right)\left(\frac{d}{dx}\right)^{n-1}\phi=0,$$

and all the values of θ that satisfy it must give the same value to the ratio $\sin n\theta : \cos n\theta$, that is to say, $\tan n\theta$. Hence the values of $n\theta$ have a common difference π, and the values of θ a common difference, that is to say, the angle between any two consecutive branches is $\frac{\pi}{n}$.

2503. (Proposed by H. R. GREER, B.A.)—Prove that the number of normals that can be drawn from a point in space of p dimensions to a (hyper)$^{p-3}$ surface (i. e., what corresponds to a *surface* in space of three dimensions) of the nth degree is $\frac{n}{n-2}\left\{(n-1)^p-1\right\}$.

Solution by the REV. J. WOLSTENHOLME, M.A.

In the same way as the number of normals is found in ordinary three

dimensions, we have, if U_p be the number drawn from a fixed point to a surface of nth degree, in hyper-geometry of p dimensions,

$$U_p = n(n-1)^{p-1} + U_{p-1},$$

the first term of the right-hand number being the number of normals parallel to one of the axes of coordinates which meet the surface at a finite distance; and the second the number which meet it at infinity. Hence we have

$$U_p - U_{p-1} = n(n-1)^{p-1}, \quad U_{p-1} - U_{p-2} = n(n-1)^{p-2}, \quad \ldots \quad U_2 = n^2;$$

therefore $U_p = n-1+(n-1)+(n-1)^2+\ldots+(n-1)^{p-1} = \dfrac{n\{(n-1)^p-1\}}{n-2}$.

[Mr. GREER sends the following *Note* on this Question:—

"Take the case of three dimensions analytically; then, taking origin as the point whence normals are drawn, and writing $\frac{dU}{dx}=A$, $\frac{dU}{dy}=B$, $\frac{dU}{dz}=C$, we have $U=0$ (equation of surface of nth degree), $\frac{x}{A}=\frac{y}{B}=\frac{z}{C}$ ....(1), or $U=0$, $xB-yA=0$, $xC-zA=0$ .....(2), and, at first sight, we should *expect* the number of solutions to be n^3. The question is, how the reduction arises?"

In reference to this point, Mr. WARREN remarks as follows:—

"Although *all* the solutions of system (1) are contained in system (2), *the reverse is not true*; for system (2) is satisfied by ($U=0$, $x=0$, $A=0$). As ($yC-zB$) is not however *in general* rendered zero by ($x=0$, $A=0$, $U=0$), we must subtract the number of solutions of ($U=0$, $A=0$, $x=0$), that is $n(n-1)$, from the number of solutions of ($U=0$, $xB-yA=0$, $xC-zA=0$), that is n^3, in order to get the number of values that satisfy the system (1), which is therefore n^3-n^2+n. For $n=2$, we get the number $=6$, which is true. The problem is a particular case of a very interesting class investigated first by CAYLEY, and extended by SALMON and S. ROBERTS."

See Frost and Wolstenholme's *Solid Geometry*, Art. 341, and Hymers' *Geometry of Three Dimensions*, p. 238, ex. 12.]

2682. (Proposed by A. W. PANTON, A.B.)—If circles be described on a pair of opposite sides of a square; prove geometrically that

1. The polars of any point on either diagonal with respect to the two circles meet on the other diagonal.
2. The four tangents from the point form a harmonic pencil.

Solution by PRINCE DE POLIGNAC.

The two parts of the question can be demonstrated separately without much difficulty. In what follows I propose to set forth their connection, and to show that both must necessarily coexist and depend upon the same general principle.

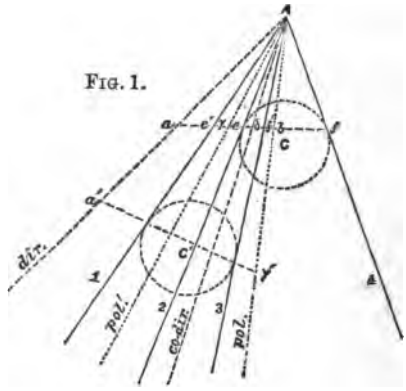
When four tangents to two conics form a harmonic pencil, the tangents to the same conics being conjugate, it is clear that, taking any pair of tangents to either conic, the pole of any one of the tangents with respect to the *other* conic lies on the *other* tangent.

This is merely another way of stating the same fact, and can be looked upon as a graphic criterion by means of which the harmonic property of the pencil can be tested.

Now it is easy to see that this criterion is only a particular case of a more general rule, which may be stated as follows; viz.,

Four lines, 1, 2, 3, 4, being given (Fig. 1), issuing from a point A; draw any two circles (or conics) C and C′ touching 2, 4 and 1, 3, respectively. Draw also through A an arbitrary line which we shall call directrix, and take its intersections a and a' with the polars of A with respect to C and to C′. Next draw the polars of a with respect to C, and of a' with respect to C′, and take their respective intersections with the polars of A with regard to C′ and to C; call them γ' and γ.

FIG. 1.



Now, if the polar of γ with respect to C coincides with the polar of γ' with respect to C′; the lines 1, 2, 3, 4 will form a harmonic pencil, and otherwise not.

Demonstration.—Let us call Co-directrix the common polar of γ and γ'. If we write *pol.* and *pol.* for the polar of a with respect to C, and that of a' with respect to C′, we shall have 8 lines issuing from A, viz.,

dir., 1, pol.′, 2, co-dir., 3, pol., 4.

Using, for shortness' sake, the notation $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ to express the harmonic ratio where the conjugate points or lines are written *under* each other, we have obviously the following harmonic pencils and corresponding points in harmonic ratio on any transversal, say the polar of A with respect to C.

$$(C)\quad \left\{ \begin{matrix} \begin{bmatrix} 2 \text{ dir.} \\ 4 \text{ pol.} \end{bmatrix} & \begin{bmatrix} e & a \\ f & b \end{bmatrix}; & \begin{bmatrix} 1 \text{ dir.} \\ 3 \text{ pol.}' \end{bmatrix} & \begin{bmatrix} e' & a \\ f' & \gamma \end{bmatrix} \\ \begin{bmatrix} 2 \text{ pol.}' \\ 4 \text{ co-dir.} \end{bmatrix} & \begin{bmatrix} e & \gamma \\ f & \delta \end{bmatrix}; & \begin{bmatrix} 1 \text{ pol.} \\ 3 \text{ co-dir.} \end{bmatrix} & \begin{bmatrix} e' & b \\ f' & \delta \end{bmatrix} \end{matrix} \right\} (C').$$

Likewise, if we use the notation $\left\{ \begin{matrix} a & c & e \\ b & d & f \end{matrix} \right\}$ to express involution, the conjugate points being written *under* each other; the harmonic system (C) will give, through well-known properties of involution,

$$\left\{ \begin{matrix} e & a & \gamma \\ e & b & \delta \end{matrix} \right\} \text{ or } \left\{ \begin{matrix} f & a & \gamma \\ f & b & \delta \end{matrix} \right\}.$$

The system (C′) gives $\begin{Bmatrix} e' & a & \gamma \\ f' & b & \delta \end{Bmatrix}$;

hence, as is well known, $\frac{e}{f}$ foci of $\begin{Bmatrix} e' & a & \gamma \\ f' & b & \delta \end{Bmatrix}$.

Therefore $\begin{bmatrix} e & e' \\ f & f' \end{bmatrix}$ or $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$;

that is to say, the four lines 1, 2, 3, 4 form a harmonic pencil.

Conversely, if 1, 2, 3, 4 form a harmonic pencil, we shall have the following four harmonic ratios:

$$\begin{bmatrix} e & a \\ f & b \end{bmatrix} \quad \begin{bmatrix} e' & a \\ f' & \gamma \end{bmatrix} \quad \begin{bmatrix} e & \gamma \\ f & \delta \end{bmatrix} \quad \begin{bmatrix} e' & e \\ f' & f \end{bmatrix}$$

From the second and first systems we draw the inferences

$$\begin{Bmatrix} e' & a & e \\ e' & \gamma & f \end{Bmatrix} \text{ and } \begin{Bmatrix} f' & a & e \\ f' & \gamma & f \end{Bmatrix}; \quad \begin{Bmatrix} e & a & b \\ f & \gamma & \delta \end{Bmatrix}.$$

Hence $\frac{e'}{f'}$ foci of $\begin{Bmatrix} e & a & b \\ f & \gamma & \delta \end{Bmatrix}$; therefore $\begin{bmatrix} e' & b \\ f' & \delta \end{bmatrix}$,

which, as in (C′), expresses that the polar of γ' with regard to circle C′ coincides with that of γ with regard to C; in other words, that a co-directrix is obtained.

If we take the co-directrix as directrix, these two lines will interchange. So will pol. and pol.′; and we can state the rule thus:—

Draw through A any directrix. Take its pole γ with regard to C. Take the polar of γ with respect to C′, and its intersection a' with the polar of A with respect to C′. Draw Aa' the co-directrix. Now resume the construction *mutatis mutandis*, *i.e.*, beginning with the same directrix and interchanging the accents, or beginning with the co-directrix without changing the accents. The same final lines will be obtained if the four tangents form a harmonic pencil, and conversely.

If one of the tangents, say 2, be taken as directrix, the harmonic symbols (C) and (C′) show that pol. must also coincide with 2, and that both pol.′ and co-dir. coincide with the conjugate tangent 4. This particular case gives the remark made at the beginning.

When the lines 1, 2, 3, 4 form a harmonic pencil, if we call A′ the point of intersection of the two polars of A, and take AA′ as directrix, we shall obtain two points γ, γ' which will determine the co-directrix. Now, if we resume the construction from A′ with the same directrix A′A, we shall obtain all the same lines in an inverted order, and the same two points γ, γ'; hence, if the polar of γ with regard to C′ coincides with the polar of γ' with regard to C, the four tangents from A′ will also form a harmonic pencil; and conversely.

In the case of the given question (Fig. 2), the middle points of the horizontal sides of the square have inversely the same two polars with regard to both circles. These polars are the two diagonals. This is sufficient to show that if a point A be taken on any diagonal, and if (1) in the given question is true for that point, (2) is true also, and *vice versâ*. Because, if we take the diagonal as directrix, and apply the last statement of the rule, the middle points of horizontal sides will be our points γ and γ', and we have to take the intersections of the polars

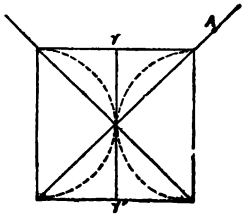


FIG. 2.

of γ and γ', *i.e.* the *second diagonal*, with the polars of A respectively. But if (1) is true for A, both intersections coincide; that is to say, we obtain a co-directrix. Hence (2) is true, and the converse holds.

Furthermore, if we take (Fig. 3) any number of points A, A_1, &c., on one diagonal, and call p, p_1, &c., p', p'_1, &c., the respective intersections of their polars with the other diagonal; if the points p, p_1 &c., do not coincide with p', p'_1, &c., they will be homographic, as to one point A corresponds only *one* point p and *one* point p', and *vice versâ*. If O, centre of the square, be taken as origin, it is obvious that when $Op = \infty$, $Op' = \infty$, and when $Op = 0$, $Op' = 0$, hence $\frac{Op}{Op'}$ = constant. Consequently, either the two conjugate points will always, or they will never, coincide.

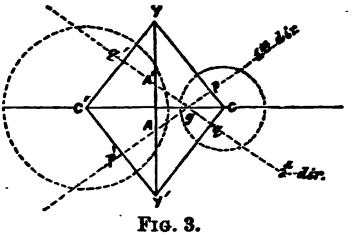


FIG. 3.

In other words,—

If (1) *or* (2) *be true for one position of* A, (1) *and* (2) *will be true for every position of* A.

Now the truth of the given question can be ascertained in various ways. For instance, A may be taken at one vertex of the square. Then its polars evidently meet at the other vertex on the common tangent. Hence, &c.

Or A may be taken at infinity, and (2) directly verified from the intercepts made by tangents 1, 3; 1, 2, and 1, 4, on the diagonal perpendicular to their direction. Taking the radius as unity, these are at once found to be 2, $\sqrt{2}$, $2+\sqrt{2}$, and obviously $\frac{2}{\text{intercept }(1,3)} = \frac{1}{(1,2)} + \frac{1}{(1,4)}$.

A converse theorem can also be proved; viz., that if the polars of A with respect to two circles meet in A′, and if the four tangents from A as well as from A′ form harmonic pencils (tangents to the same circles being conjugate), the circles must be equal and must also touch each other.

It follows from what has been said, that in this case there must be in the plane two such points as γ, γ' whose polars with respect to the two circles are inversely coincident. Now it is clear that the figure $C\gamma C'\gamma'$ must be a parallelogram (Fig. 3), and if we call p, q the distances from C to the two polars, and the same accented from C′, and put

$$CC' = c, \quad C\gamma = l, \quad C\gamma' = l', \quad C'\hat{C}\gamma = \gamma, \quad C'\hat{C}\gamma' = \gamma',$$

we have at once $pl = r^2$, $ql' = r^2$, $p'l = r'^2$, $q'l' = r'^2$ ..... (1, 2, 3, 4);

hence $\frac{p}{p'} = \frac{q}{q'}$. This result shows that the points of intersection of $\overline{CC'}$ with the two polars either coincide or divide CC′ harmonically. But it is easy to see that they must coincide, from the consideration that the centre can never lie between the pole and its polar. Therefore, calling g the distance from C to the intersection of the two polars, we have the additional equations

$$\frac{p}{p+p'} = \frac{g}{c}, \quad p = g\cos\gamma, \quad q = g\cos\gamma' \ldots\ldots\ldots (5, 6, 7),$$

$$l\cos\gamma + l'\cos\gamma' = c, \quad l\sin\gamma = l'\sin\gamma' \ldots\ldots\ldots\ldots (8, 9).$$

(1) and (2) give $pl = ql'$, which, combined with (6), (7), and (9), leads to

$$\gamma = \gamma', \quad l = l', \quad p = q, \quad c = 2l\cos\gamma;$$

and the former equations reduce to

$$pl = r^2, \quad p'l = r'^2, \quad p + p' = c\cos\gamma, \quad c = 2l\cos\gamma \ldots\ldots (1\ldots4).$$

(1), (2), and (3) give at once $r^2 + r'^2 = cl\cos\gamma$, which, combined with (4), gives $c^2 = 2(r^2 + r'^2) = (r+r')^2 + (r-r')^2$, therefore $c > r + r'$.

If r and r' be given and c determined from the above equation, the circles cannot meet in real points, and a locus will be obtained for γ and γ', viz., the perpendicular drawn to the line joining centres half-way between them. γ and γ' will be equidistant from the line joining centres, and it is clear that the point at which the polars of γ and γ' meet has the line $\gamma\gamma'$ for polar with respect to both circles.

Now two points γ, γ' being selected, if two points A, A′ can be found whence the tangents form harmonic pencils, they must lie on the polars of γ and γ', each of which here becomes the directrix (or co-directrix). And it will be seen as above that, taking any number of points on the 1st directrix, the intersections of their polars with the 2nd directrix will be homographic, so that A′ is the point which is its own conjugate, and the same for A. But these points are in both systems, the point (g) where the two directrices meet, and the points A and A′ where these lines meet γ, γ'. If γ and γ' are taken outside the circle, A and A′ lie inside; if γ, γ' are taken inside the circle, the tangents which should be conjugates lie outside of the other two; therefore no harmonic pencils can be obtained, except in the limiting case where $r = r'$, as then $c = r + r'$; that is to say, the circles are equal and touch each other.

It will also be seen that, in the general case, although A and A′ are *real* points, the harmonic symbols in the systems (C) and (C′) reduce to two, therefore no inference can be drawn.

2748. (Proposed by W. K. Clifford, B.A.)—If a circular cubic with a double point O be cut by a circle in four points A, B, C, D; and if OA, OB, OC, OD cut the circle again in E, F, G, H; show that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O.

I. *Solution by* Archer Stanley.

This theorem is the inverse of the following well known one:—

Any two opposite connectors of the four points common to a circle and a conic are equally inclined to the axes of the latter.

For if the double point O of the circular cubic be taken as the centre of inversion, the inverse curve will be a conic passing through O, whose asymptotes are parallel to the tangents at the double point, and whose axes, therefore, are parallel to the bisectors of the angles made by these tangents. Again, the inverse of the circle cutting the cubic in A, B, C, D will be another circle cutting the conic in A′, B′, C′, D′, and the points E′, F′, G′, H′, respectively inverse to E, F, G, H, will of course be the second intersections of A′O, B′O, C′O, D′O with the circle A′B′C′D′.

Now, by a well known property of two inverse circles, the connector of any two of the four points E, F, G, H is parallel to that of the corresponding

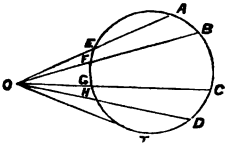
two of the four points A′, B′, C′, D′. (For instance, EF and A B , being each anti-parallel to AB, are parallel to one another). The truth of the proposed theorem is consequently obvious.

II. *Solution by* W. S. McCay, B.A.

The inverse of the cubic with respect to the double point is a conic, whose asymptotes are parallel to the tangents at the double point. The first part is evident from the form of the equation

$$(x^2+y^2)u_1+ax^2+2hxy+by^2=0,$$

the origin being at the double point; and the second part follows.



Clearly E, F, G, H are points on the inverse curve if $\rho\rho' = OT^2$; therefore E, F, G, H are the intersections of the circle with a conic whose axes are parallel to the bisectors of angles between tangents at the double point O.

It appears in the same way that the theorem is equally true for a bicircular quartic; for the curve $(x^2+y^2)^2+(x^2+y^2)u_1+ax^2+2hxy+by^2=0$ also inverts into a conic, the origin not being a point on the *conic*, as it is in the other case.

2718. (Proposed by Professor Cayley.)—Find *in plano* the locus of a point P, such that from it two given points A, B, and two given points, C, D subtend equal angles.

I. *Solution by* W. S. Burnside, M.A.

1. Let the length of AB (Fig. 1) be $2a$, and its equation $\alpha=0$; the length of the bisector PM of the base ρ, and the angle $APB=\theta$. Then we have

$$AB^2 = AP^2+BP^2-2AP.BP\cos\theta,$$

or $AP.BP\cos\theta=\rho^2-a^2$, also $AP.BP\sin\theta=2a\alpha$;

hence $$\tan\theta=\frac{2a\alpha}{\rho^2-a^2}.$$

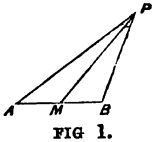


FIG 1.

Replacing ρ^2-a^2 by U, where $U=0$ is the equation of the circle on AB as diameter, $\tan\theta=\frac{2a\alpha}{U}$. Similarly, if CD subtends an angle η, $\tan\eta=\frac{2b\beta}{V}$. Hence, if $\theta=\eta$, the equation of the locus is $a\alpha V=b\beta U$, which represents a circular cubic. (See Salmon's *Higher Plane Curves*, Art. 141.)

2. In a similar manner, we may investigate the locus of P, when the points P, A, B, C, D are situated on the surface of a sphere (Fig. 2).

In this case we have

$$\tan\theta=\frac{\sin 2a.\alpha}{\sin^2\rho(1-\sin^2 a\sin^2\alpha)-\sin^2\alpha},$$

where ω is the angle which the bisector ρ of AB makes with AB, and $AB=2a$ as before. Hence we may write $\tan\theta=\frac{\sin 2a.\alpha}{U}$, where $U=0$ is

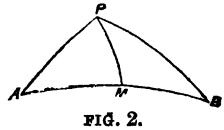


FIG. 2.

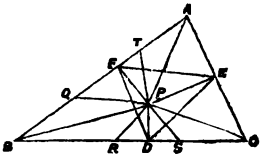
the equation of a sphero-conic. And we have for the equation of the locus

$$\sin 2a \,.\, \alpha V = \sin 2b \,.\, \beta U$$

(where α is the sine of the arc through P perpendicular to AB, or $\alpha = 0$ the equation of the plane through AB and the centre of the sphere).

II. *Solution by the* EDITOR.

1. Let BC ($=a$) and BA ($=c$) be two given lines meeting at a given angle at B, and let P be a variable point at which BC and BA subtend equal or supplementary angles; then, drawing PD, PE, PF perpendicular to BC, CA, AB, we have



$$\frac{PD}{PF} = \frac{\sin PBD}{\sin PBF} = \frac{\sin PBC}{\sin BPC} \cdot \frac{\sin BPA}{\sin PBA}$$

$$= \frac{PC}{BC} \cdot \frac{BA}{PA} = \frac{PC \,.\, \sin BCA}{PA \,.\, \sin BAC} = \frac{DE}{EF},$$

therefore

$$PD^2 \,.\, EF^2 = PF^2 \,.\, DE^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now, taking ABC as triangle of reference, let the trilinear coordinates of P be (x, y, z); then the relation (1) gives at once for the equation of the locus of P

$$x^2(y^2 + z^2 + 2yz \cos A) = z^2(x^2 + y^2 + 2xy \cos C) \ldots\ldots\ldots (2).$$

Reducing (2), we find that part of the locus is

$$y = 0, \text{ that is to say, the line AC} \quad \ldots\ldots\ldots\ldots\ldots (3),$$

as indeed might have been seen at once without calculation, since BA and BC obviously subtend the same angle at a point in AC produced, and supplementary angles at any point in AC between A and C.

The other part of the locus is the cubic

$$(x^2 - z^2)\, y + 2xz\,(x \cos A - z \cos C) = 0 \quad \ldots\ldots\ldots\ldots (4),$$

which may be written in the form $xV - zU = 0$,

where $U \equiv yz + x(-x \cos A + y \cos B + z \cos C)$,

and $V \equiv xy + z(\ x \cos A + y \cos B - z \cos C)$;

so that $U = 0$, $V = 0$ are the equations of the circles on BC, BA as diameters.

If the angles A and C are equal, or BA = BC, the cubic locus (4) breaks up into the line $x - z = 0$, that is, the bisector of the angle B, and

$$\frac{1}{x} + \frac{2 \cos A}{y} + \frac{1}{z} = 0, \quad \text{or} \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0,$$

which is the equation of the circle circumscribing the triangle ABC.

2. Next suppose CR ($=k$), and AQ ($=k$) to be the fixed lines which subtend equal angles at P; then, since

$$\cot RPC = \cot(RPD + DPC) = \frac{PD^2 - RD \,.\, DC}{PD \,.\, RC},$$

we have, when the angle RPC = QPA, the relation

$$AQ \,.\, PF\,(PD^2 - RD \,.\, DC) = CR \,.\, PD\,(PF^2 - AF \,.\, FQ) \quad \ldots\ldots\ldots (5)$$

The bracketed expressions in (5) are those (U, V, say) which when equated to zero give the equations of the circles on CR, AQ as diameters; hence, for supplementary or equal values of the angles CPR, ABQ, the locus of P is the circular cubic

$$hx\mathrm{V} \pm kz\mathrm{U} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

We may express U in various ways: thus, if S, T be the middle points of CR, AQ, we have $\mathrm{RD}.\mathrm{DC} = \mathrm{SC}^2 - \mathrm{SD}^2$, and

$$\mathrm{U} \equiv \mathrm{PD}^2 - \mathrm{RD}.\mathrm{DC} \equiv \mathrm{PC}^2 - \mathrm{RC}.\mathrm{CD} \equiv \mathrm{SP}^2 - \mathrm{SC}^2 \ldots\ldots\ldots (7).$$

If we take, as before, ABC as triangle of reference, and put (x, y, z) for the trilinear coordinates of P, we have

$$\mathrm{CD} = x\cot \mathrm{C} + y \operatorname{cosec} \mathrm{C}, \text{ and } \mathrm{CP}^2.\sin^2 \mathrm{C} = \mathrm{DE}^2 = x^2 + y^2 + 2xy\cos \mathrm{C},$$

therefore $\mathrm{U} \equiv (x^2 + y^2 + 2xy\cos \mathrm{C})\operatorname{cosec}^2 \mathrm{C} - h(x\cot \mathrm{C} + y \operatorname{cosec} \mathrm{C}) \ldots\ldots (8).$

From this we may at once write down the trilinear equation of the locus, and render it homogeneous, if necessary. Under another form the equation of the locus is given in the Solution of Question 1721, on p. 41 of Vol. IV. of the *Reprint*.

3. If the two given lines are in the same straight line, BR and CS in the above figure suppose, the locus of P is a circle having as the ends of a diameter the two points which divide BC internally and externally in the ratio of BM : CN, BM and CN being tangents from B and C to a circle on RS as diameter. This may be very simply shown as follows:—

$$\frac{\mathrm{BP}.\mathrm{PS}}{\mathrm{CP}.\mathrm{PR}} = \frac{\triangle \mathrm{BPS}}{\triangle \mathrm{CPR}} = \frac{\mathrm{BS}}{\mathrm{CR}}, \text{ and } \frac{\mathrm{BP}.\mathrm{PR}}{\mathrm{CP}.\mathrm{PS}} = \frac{\triangle \mathrm{BPR}}{\triangle \mathrm{CPS}} = \frac{\mathrm{BR}}{\mathrm{CS}},$$

therefore

$$\frac{\mathrm{BP}^2}{\mathrm{CP}^2} = \frac{\mathrm{BR}.\mathrm{BS}}{\mathrm{CR}.\mathrm{CS}} = \frac{\mathrm{BM}^2}{\mathrm{CN}^2};$$

hence we have

$$\mathrm{BP} : \mathrm{CP} = \mathrm{BM} : \mathrm{CN},$$

and therefore the locus of P is the above-mentioned circle.

2737. (Proposed by Professor CAYLEY.)—Find *in solido* the locus of a point P, such that from it two given points, A, C, and two given points B, D, subtend equal angles.

Solution by J. J. WALKER, M.A.

By steps similar to those adopted in the Solution of the preceding Question (2718), the equation of the locus is at once found to be

$$a^2\left[\left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\} - \left\{l(x-x') + m(y-y') + n(z-z')\right\}^2\right] \times \left[(x-x'')^2 + (y-y'')^2 + (z-z'')^2 - b^2\right]^2 =$$

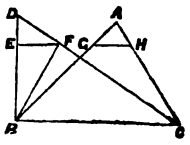
$$b^2\left[\left\{(x-x'')^2 + (y-y'')^2 + (z-z'')^2\right\} - \left\{l'(x-x'') + m'(y-y'') + n'(z-z'')\right\}^2\right] \times \left[(x-x')^2 + (y-y')^2 + (z-z')^2 - a^2\right]^2,$$

where (x', y', z') are the coordinates of the middle point of AB, and (l, m, n) its direction cosines, (x'', y'', z'') and (l', m', n') referring similarly to CD, and (x, y, z) being the coordinates of the point whose locus is sought—a surface of the sixth degree.

2485. (Proposed by W. HOPPS.)—In an acute-angled scalene triangle to inscribe the rectangle of minimum diagonal, having one of its sides coincident with a side of the triangle.

Solution by T. DOBSON, B.A.; R. TUCKER, M.A.; W. H. LAVERTY; *the* PROPOSER; *and many others.*

In the triangle ABC draw BD perpendicular to BC and equal to the perpendicular from A on BC, and join CD.



Draw *any* right line parallel to BC, cutting BD, CD in E and F, and AB, AC in G and H. Then

$$\frac{\triangle DEF}{\triangle DBC} = \frac{DE^2}{DB^2} = \frac{\triangle AGH}{\triangle ABC};$$

but $\triangle DBC = \triangle ABC$; $\therefore \triangle DEF = \triangle AGH$; and $\therefore EF = GH$.

Hence the rectangle on the diagonal BF, and the inscribed rectangle of which GH is a side, are identically equal; and it is obvious that BF is a minimum when at right angles to CD. Hence this

Construction.—Draw BF perpendicular to CD, and from F draw perpendiculars to BC, BD; then the rectangle on diagonal BF is evidently equal to the rectangle of minimum diagonal inscribed in ABC, and having a side in BC.

COR.—If $BC = a$, $BD = p_1$, $BF = u_1$; we have $u_1^{-2} = a^{-2} + p_1^{-2}$.

[From Mr. DOBSON'S *Corollary* we readily obtain

$$u_1^{-2} - u_2^{-2} = (b^{-2} - a^{-2}) \cot^2 C;$$

whence we see that the rectangle on the *longest* side of the triangle will have the *shortest* diagonal.]

2612. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If points P, Q, R be taken at random, one on each of the sides BC, CA, AB, of the triangle ABC, the chance that the area of the triangle PQR shall be greater than $(n + \frac{1}{4})$ of the area of the triangle ABC (n being positive and $< \frac{3}{4}$) is

$$\frac{3-4n}{4} - \frac{12n+1}{4} \log \frac{4}{4n+1} + 8n^{\frac{3}{2}} \left(\tan^{-1} \frac{1}{2n^{\frac{1}{2}}} - \frac{\pi}{6} \right).$$

I. *Solution by the* PROPOSER.

Taking $BP = ax$, $CQ = by$, $AR = cz$, the area Δ' of the triangle PQR, in parts of the area of ABC, is

$$\Delta' = 1 - z(1-y) - x(1-z) - y(1-x) = 1 - x - y - z + yz + zx + xy,$$

or, putting $x = X + \frac{1}{2}$, &c., we have $\Delta' = \frac{1}{4} + YZ + ZX + XY$; hence the chance required is the chance that $YZ + ZX + XY$ shall be greater than n; X, Y, Z being restricted within the limits $\pm \frac{1}{2}$, and all values within those limits equally probable. This amounts to finding the volume common to a

hyperboloid of revolution of two sheets (when n is positive) and to a cube of volume unity whose diagonal coincides with the axis of the hyperboloid and whose faces are equally inclined to that axis. It is easily seen that this is six times the volume included within the hyperboloid and the planes $Z=X$, $Z=Y$, $Z=\frac{1}{2}$. But the area of hyperbola $(X+z)(Y+z)=z^2+n$ included between the straight lines $X=z$, $Y=z$, is $3z^2-n-(z^2+n)\log\frac{4z^2}{z^2+n}$, hence the volume required is $6\int_{(\frac{1}{2}n)^{\frac{1}{2}}}^{\frac{1}{2}}\left\{3z^2-n-(z^2+n)\log\frac{4z^2}{z^2+n}\right\}dz$,

the lower limit being the value of z at the vertex of the hyperboloid. The integration presents no difficulty, and gives the result as above.

Hence, if $n=0$, the chance that $\Delta'>\frac{1}{4}$ is $\frac{1}{2}-\frac{1}{2}\log 2$,
$n=\frac{1}{4}$, the chance that $\Delta'>\frac{1}{4}$ is $\frac{1}{4}-\log 2+\frac{1}{12}\pi$,
$n=\frac{3}{4}$, the chance that $\Delta'>1$ is 0, as it obviously ought to be.

[A Solution of the case when $n=\frac{1}{4}$ or $\Delta'>\frac{1}{8}$ was given by the EDITOR under Quest. 1230 in the *Educational Times* for October, 1862, and on p. 96 of Vol. II. of the *Reprint*.]

II. *Solution by* STEPHEN WATSON.

Let L, M, N be the middle points of BC, CA, AB, and put $PL=ax$, $MQ=by$, $NR=cz$; then we readily find that the area Δ' of the triangle PQR, in parts of that of ABC, is

$$\Delta'=1-(\tfrac{1}{2}-x)(\tfrac{1}{2}+z)-(\tfrac{1}{2}+x)(\tfrac{1}{2}-y)-(\tfrac{1}{2}+y)(\tfrac{1}{2}-z)$$
$$=\tfrac{1}{4}+xy+yz+zx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

When PQR is to be $>\frac{1}{4}$ABC, or $\Delta'>\frac{1}{4}$, (2) becomes

$$xy+yz+zx>0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which is fulfilled for all positive values of x, y, z; also for all negative values when x, y, z are *all* negative.

The chance in either case is $\frac{1}{8}$, and therefore of both $=\frac{1}{4}$ $\ldots\ldots\ldots$ (3).

When x is negative, or yz are *both* negative, in either case (2) gives $x<\frac{yz}{y+z}$; hence, in both cases, we have

$$2\int_0^{\frac{1}{2}}dy\int_0^{\frac{1}{2}}\frac{yz}{y+z}\,dz=\int_0^{\frac{1}{2}}\left(y-2y^2\log\frac{y+\frac{1}{2}}{y}\right)dy=\tfrac{1}{8}-\tfrac{1}{4}\log 2 \ldots\ldots (4).$$

The result is the same when y or zx is negative, and also when z or xy is negative, and these include all possible favourable cases; hence, putting p_1 for the probability in this case, we have

$$p_1=(3)+3(4)=\tfrac{3}{4}-\tfrac{1}{2}\log 2.$$

When PQR is to be $>\frac{1}{2}$ABC, or $\Delta'>\frac{1}{2}$, (2) becomes

$$xy+yz+zx-\tfrac{1}{4}<0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5);$$

therefore $z>\frac{\frac{1}{4}-xy}{x+y}$; and since this must be $<\frac{1}{2}$, therefore $y>\frac{\frac{1}{2}-x}{1+2x}$.

Also it is plain from (6) that neither one nor two of the quantities x, y, z can be negative, but all three may; hence, doubling on account of this, and putting p_2 for the probability in this case, we have

$$p_2 = 2\int_0^{\frac{1}{2}} dx \int_{\frac{\frac{1}{2}-x}{1+2x}}^{\frac{1}{2}} \frac{\frac{1}{4}-xy}{x+y}\,dy$$

$$= 2\int_0^{\frac{1}{2}} \left\{x + (x^2+\tfrac{1}{4})\log\frac{x^2+\frac{1}{4}}{(x+\frac{1}{2})^2}\right\} dx$$

$$= \frac{1}{2} + \frac{\pi}{12} - \log 2.$$

2704. (Proposed by J. J. WALKER, M.A.)—A central conic $(a^2y^2 \pm b^2x^2 = a^2b^2)$ is turned about its centre in its own plane through a right angle; prove that the locus of the intersection of any tangent to the given conic with the same line in its new position is one of the two quartics

$$(x^2+y^2)^2 = a^2(x \mp y)^2 \pm b^2(x \pm y)^2.$$

I. *Quaternion Solution by* W. H. LAVERTY, B.A.

The vectors to the two tangents will evidently be

$$\rho = \quad a\cos\tau + \beta\sin\tau + x(-a\sin\tau + \beta\cos\tau),$$

and $$\rho' = -a\sin\tau + \beta\cos\tau + y(\ a\cos\tau + \beta\sin\tau).$$

At their intersection $\rho = \rho'$, and therefore $x = y = 1$, and we get for the equation to the locus

$$\rho = a(\cos\tau - \sin\tau) + \beta(\sin\tau + \cos\tau);$$

therefore $\tau^2\rho = \tau^2 a(\cos\tau - \sin\tau)^2 + \tau^2\beta(\cos\tau + \sin\tau)^2$, since $S\alpha\beta = 0$;

therefore $\tau^4\rho = a^2(\tau\rho\,.\cos\tau - \tau\rho\,.\sin\tau)^2 + b^2(\tau\rho\,.\cos\tau + \tau\rho\,.\sin\tau)^2$;

or $$(x^2+y^2)^2 = a^2(x-y)^2 + b^2(x+y)^2.$$

Similarly for the hyperbola.

II. *Solution by the* PROPOSER; REV. J. L. KITCHIN; *and others.*

Taking b^2 positive, and supposing the direction of rotation to be from the positive side of the axis of x towards the negative side of the axis of y, if the equation to any given tangent be $a^2y'y + b^2x'x = a^2b^2$, that of the same line in its new position must be $a^2y'x - b^2x'y = a^2b^2$, since these two lines are at right angles to one another and cut off equal intercepts, measured from the centre, on *different* axes. The coordinates of the point of intersection of these lines are

$$(a^4y'^2 + b^4x'^2)x = a^2b^2(a^2y' + b^2x') \text{ and } (a^4y'^2 + b^4x'^2)y = a^2b^2(a^2y' - b^2x') \ldots (1).$$

Squaring and adding these equations, and dividing by $a^4y'^2 + b^4x'^2$,

$$(a^4y'^2 + b^4x'^2)(x^2+y^2) = 2a^4b^4 \ldots\ldots\ldots\ldots\ldots (2).$$

Multiplying equations (1),

$$(a^4y'^2 + b^4x'^2)^2 xy = a^4b^4(a^4y'^2 - b^4x'^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

But $a^4y'^2 + b^4x'^2 = b^2\{a^4 - (a^2 - b^2)x'^2\}$ and $a^4y'^2 - b^4x'^2 = b^2\{a^4 - (a^2 + b^2)x'^2\}$.

Hence (2) becomes $(a^2 - b^2)(x^2 + y^2)x'^2 = a^4(x^2 + y^2 - 2b^2)$,
and (3) divided by (2) gives

$$\{a^2(x-y)^2 + b^2(x+y)^2\}x'^2 = a^4(x-y)^2.$$

Eliminating x'^2, there results for the required locus

$$(a^2 - b^2)(x^2 + y^2)(x-y)^2 = (x^2 + y^2 - 2b^2)\{a^2(x-y)^2 + b^2(x+y)^2\},$$

which, after reduction, becomes

$$(x^2 + y^2)^2 = a^2(x-y)^2 + b^2(x+y)^2.$$

If the rotation had been in the opposite direction, the equation to the tangent in its new position would have been $b^2x'y - a^2y'x = a^2b^2$, and the sign of y would have been changed in the corresponding locus.

Note on Analagmatic Pavements, Quest. 2552; *by* Professor Sylvester.

(*See diagram opposite.*)

In the Analagmatic Pattern or Pavement here drawn (so called from the relation of the lines or columns to one another remaining unaltered as we pass from one to another) it will be observed that each line or column consists of 6 squares of one colour and 10 of the contrary colour: there being 8 lines or columns of the kind where black preponderates, and 8 where white preponderates.

In general, a pattern containing 2^{2m} (say n) compartments in each of its lines and columns may be constructed not only analagmatic, but so as to satisfy a similar condition to the aboye, viz., such that half of the lines and columns shall each contain $\frac{1}{2}(n + \sqrt{n})$ white and $\frac{1}{2}(n - \sqrt{n})$ black, while each of the remaining lines and columns contain $\frac{1}{2}(n - \sqrt{n})$ white and $\frac{1}{2}(n + \sqrt{n})$ black compartments. Pavements of this character, where the difference between the colours in each line and column is constant, may be termed isochromatic. It is easy to prove that analagmatic pavements cannot be made isochromatic when n is an odd power of 2.

The construction of analagmatic synchromatic squares it was which led the author to the discovery of the transcendental property of the binomial coefficients given in Question 2802 of the *Educational Times*.

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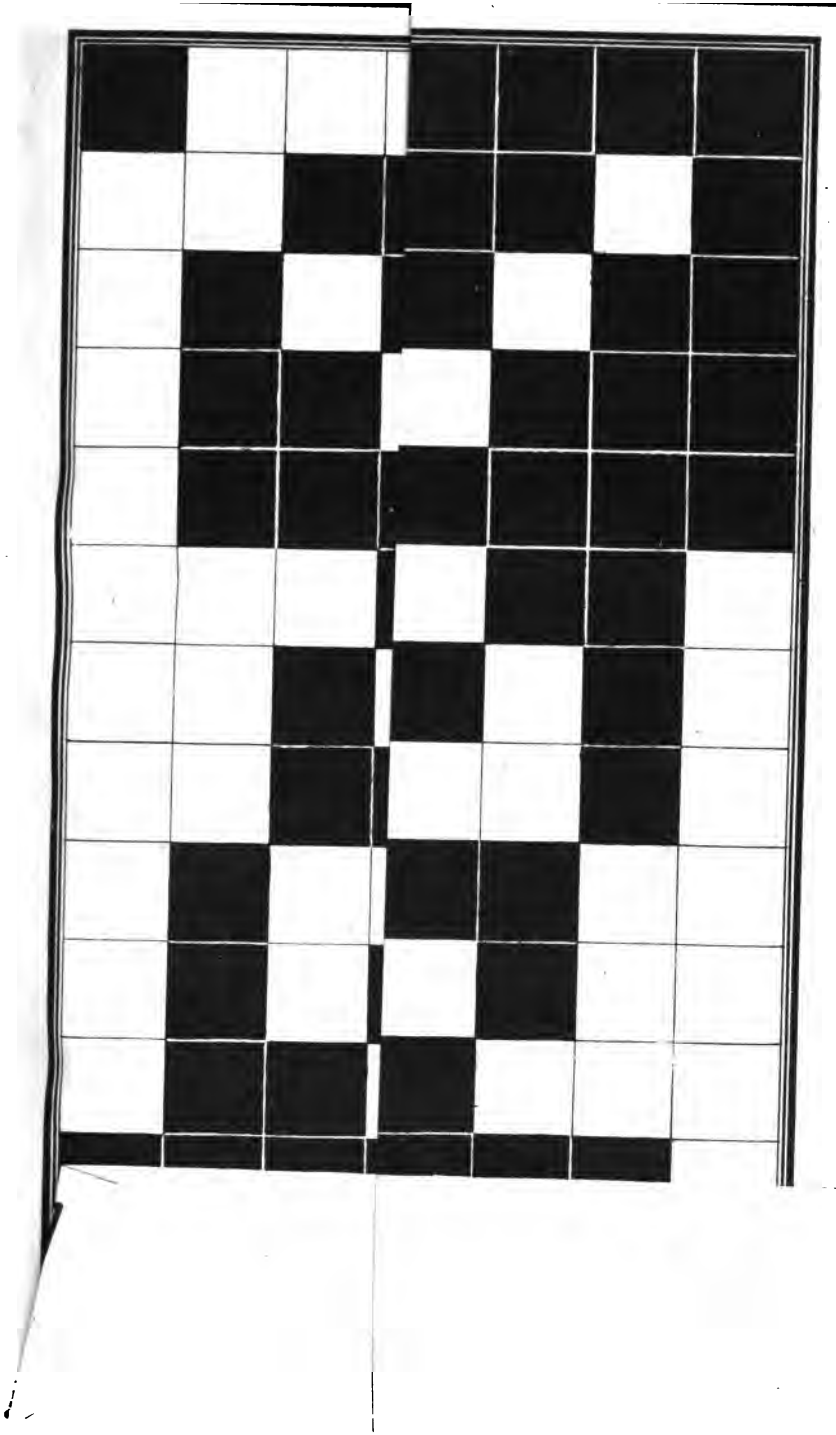

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